ON THE SIMILARITY SOLUTIONS OF THE SEMILINEAR HYPERBOLIC EQUATION $u_{xt} = f(u)$ VIA SYMMETRY METHOD.

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ABSTRACT

Herein the physically and mathematically singnificant semilinear hyperbolic equation $u_{xt} = f(u)$, *where f(u) is an arbitrary smooth function of u and which encompasses Liouville equation, phi-four equation, Sine-Gordon equation, Klein-Gordon equation, Mikhailov equation and Double Sine-Gordon equation, has been analysed via the symmetry method as developed by Steinberg [1]. The infinitesimals, similarity variables, dependent variables and reduction to quadrature or exact solutions have been tabulated for the mentioned physical forms of f(u). Some interesting outcomes of this study are the deduction of the new exact solutions that does not seem to have been reported in the literature.*

INTRODUCTION

Ever since the work of Birkhoff [2] on " the method of search for symmetric solutions " the technique of similarity transformation has turned out to be one of the most powerful techinques for solving non-linear differential equations. Its application has resulted in a large number of solutions of the equations originating mostly in the realm of continuum mechanics. Nevertheless, the technique could not become as popular as the corresponding technique of variable separable form and Laplace transform technique for solving linear differential equations, this could be due to the following reasons [3,4] :-

(i) The main apparatus of the Lie group theory concept could not be made algorithmic.

(ii) The Lie group analysis based on transformation theory of differential equations, though a powerful and systematic approach, for obtaining the similarity solutions of linear or nonlinear, ordinary or partial differential equations could not be easily exploited for differential equations of higher order and for a system of partial differential equations which is must for most of the physical and engineering situations.

(iii) No link could be established with the corresponding technique of variable separable form for solving linear differential equations.

Consequently, an attempt had to be made for devising a technique which could take care of all the above features and yield many more new solutions than the conventional similarity transformation technique. One such technique is that of symmetry method due to Steinberg (1979). Undoubtedly, the technique involves very sophisticated tools of the theory of non-linear operators. Yet, it has been cast in a form that is easy to utilize by specialists or non-specialists alike.

Herein, we have utilized the symmetry method to generate new similarity solutions of important class of equations-semilinear hyperbolic equations. This has resulted in, for physically realizable forms of the function of dependent variable involved, a number of new solutions of the corresponding differential equations, either by reducing them to standard Painleve forms or by solving them exactly. Some interesting outcomes of this study are the deductions of new exact solutions of Liouville equation, Sine-Gordon equation, Mikhailov equation, Double Sine-Gordon and the generalized form of the Klein-Gordon equation.

SYMMETRY METHOD

As pointed out above herein we briefly outline Steinberg's (1979) similarity method of finding explicit solutions of both linear and nonlinear partial differential equations. The method is based on finding the symmetries of the differential equation and is as follows:-

Suppose that the differential operator L can be written in the form:

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u). \qquad (2.1)$$

where u=u(x,t) and H may depend on x,t,u and any derivative of u as long as the derivative of u *does not contain more than* p-1, t derivatives. Consider the symmetry operator called infinitesimal symmetry, which being quasilinear partial differential operator of first order, have the form:

$$S(u) = A(x,t,u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(x,t,u) + C(x,t,u). \qquad (2.2)$$

Define the Frechtet derivative of L(u) by

$$F(L,u,v) = \frac{d}{d\varepsilon} \quad L(u+\varepsilon v) \quad \Big|_{\varepsilon=0} \qquad (2.3)$$

with these definitions in mind we need to follow the following steps:

(i) Compute F(L, u, v).

(ii) Compute F(L, u, s(u)).

(iii) Substitute H(u) for $\frac{\partial^p u}{\partial t^p}$ in F(L, u, S(u)).

(iv) Set this expression to zero and perform a polynomial expansion.

(v) Solve the resulting partial differential equations.

Once this resulting system of partial differential equations is solved for the coefficients of S(u), equation (2.2) can be used to obtain the function form of the solution.

SEMILINEAR HYPERBOLIC EQUATION

We consider the semilinear hyperbolic equation of the form

$$L(u) = u_{xt} - f(u) = 0, \tag{3.1}$$

where f(u) is an aribitrary function of u. For the case $f(u) = e^{nu}$, $n \neq 0$ equation (3.1) becomes a field theoretic model [5]. For $f(u) = u^3 - u$, equation (3.1) serves as a model of nonlinear meson theory of nuclear forces [6] and in nonlinear theory of elementary particles [7]. For the case $f(u) = k_0 \,\mathrm{Sin}\, u$, where k_0 is a constant, equation (3.1) finds applications in many areas of physics and mathematical sciences including a coustics with special reference to propagation in ferromagnetic materials of waves carrying rotations of direction of magnetization, propagation of ultra short optical pulses, propagation in a large Josephian junctions and propagation of crystal dislocation, magnetic flux [8]. Further, equation (3.1) has been studied extensively for geometric properties [9], for Bachlund transformations [10], for Painleve analysis by many authors in the recent past, and particularly so by Clarkson et al [11] for the case $f(u) = C_1 e^{\beta_0 u} + C_2 e^{2\beta_0 u} + C_3 e^{-\beta_0 u} + C_4 e^{-2\beta_0 u}$ where C_1, C_2, C_3, C_4 and

β_0 are arbitrary constants and finally for isovector approach by Bhutani et al [12] and new similarity method in [13].

4-DETERMINATION OF THE SOLUTION

In order to find the symmetries of the equation (3.1), we set:

$$S(u) = A(x, t, u)\frac{\partial u}{\partial t} + B(x, t, u)\frac{\partial u}{\partial x} + C(x, t, u) \quad . \qquad (4.1)$$

Calculating the Frechet derivative F(L, u, ψ) of L(u) in the direction of ψ, given by equation (3.1), and replacing ψ in F by S(u), we get,

$$F(L, u, S(u)) = [S(u)]_{xt} - f'(u)\,[S(u)]. \qquad (4.2)$$

Substituting the values of different derivatives of S(u) in the resulting eqaution and calculating the coefficients of various powers of the derivatives of u in F, we get a polynomial expansion in u_x, u_t, $u_x u_t$, ... etc. On making use of equation (3.1) in the polynomial expression for F, rearranging the terms of various powers of derivatives of u and equating them to zero, we arrive at the following equations (see Appendix A).

$$\left.\begin{array}{l} A=A(t),\ \ B=B(x) \\ C_{uu}=0,\ \ C_{ut}=0,\ \ C_{ux}=o, \\ C_{xt}-f'(u)C+(A_t+B_x+C_u)f(u)=0 \end{array}\right\} \qquad (4.3)$$

On solving the system (4.3) for arbitrary form of the function f(u) we find,

$$A = A(t),\ \ B = B(x),\ \ C = C_1(x,t), \qquad (4.4)$$

and

$$C_{1\,xt} - f^{\prime}(u)\, C_1 + (A_t + B_x)\, f(u) = 0 \qquad (4.5)$$

In solving the system (4.4) - (4.5), we confine our attention to the physically interesting situations wherein f(u) represent Liouville equation, Phi-four equation, Sine-Gordon equation, Klein-Gordon equation, Mikhailov equation, and Double Sine-Gordon equation. We give below in tabular form, six different values of A, B, and C satisfying the system of equations (4.4) - (4.5) for each of the six different forms of the function f(u) and their corresponding invariants.

5. SIMILARITY SOLUTIONS AND REDUCTION TO PAINLEVE FORMS:

Herein, we utilize the similarity variable ξ and the corresponding form of u tabulated in the above section for obtaining the ordinary differential equations for $\eta(\xi)$, for six different forms of the Semilinear hyperbolic equation, given below.

5.1 Liouville Equation $f(u) = e^{nu}$, $n \neq 0$

Case I (Table 1, row 1). For the invariant transformation corresponding to case under consideration, Liouville equation is reduced to the following ordinary differential equation for $\eta(\xi)$.

$$\eta\eta^{\prime\prime} - \eta^{\prime 2} - n\,\eta = 0. \qquad (5.1.1)$$

Using the substitution

$$\eta^{\prime} = p\,, \qquad (5.1.2)$$

equation (5.1.1) gets transformed to

$$pp^{\prime} + f(\eta)p^2 - n = 0, \qquad (5.1.3)$$

where $\quad f(\eta) = -\frac{1}{\eta}$.

The solution to equation (5.1.3) can be expressed as

$$p^2 e^{2\gamma(\eta)} = C + n\int e^{2\gamma(\eta)} d\eta, \tag{5.1.4}$$

where

$$\gamma(\eta) = \int f(\eta) d\eta, \tag{5.1.5}$$

and C is a constant of integration.

Using the expression for $f(\eta)$ in equation (5.1.5), the equation (5.1.5) can be expressed as

$$p^2 = C\eta^2 - 2n\eta. \tag{5.1.6}$$

For the solution of equation (5.1.6) two possibilities arise

Case (i): C = 0

Corresponding to this possibility $\eta(\xi)$ can be expressed as

$$\eta(\xi) = -\tfrac{1}{2}n(\xi + \xi_0)^2, \tag{5.1.7}$$

where ξ_0 is a constant of integration.

Hence the required solution of Liouville equation is obtained as

$$u(x, t) = \frac{1}{n}\log\left[\frac{-\frac{2}{n}}{\gamma(t)\beta(x)(\phi(x) - \theta(t) + \xi_o)^2}\right], \tag{5.1.8}$$

where ϕ (x) and θ(t) are arbitrary functions of their respective arguments. It may be mentioned that equation (5.1.8) represents a known general solution to Liouville equation that coincides with the one obtained by Tamizhmani and Lakshmanan (1986) via Painleve analysis [14] and via new similarity technique [13] when n=1.

Case (ii): C ≠ 0

Corresponding to this possibility $\eta(\xi)$ satisfies

$$\eta(\xi) = \frac{2n}{C\left[Sech^2 \frac{\sqrt{C}}{2}(\xi + \xi_1)\right]}, \tag{5.1.9}$$

where ξ_1 another constant of integration.

Hence we get the following solution to Liouville equation. It may

$$u(x,t) = \frac{1}{n}\log\left[\frac{C\,Sech^2\frac{\sqrt{C}}{2}(\Phi(x)-\theta(t)+\xi_1)}{2n\alpha(t)\beta(x)}\right]. \qquad (5.1.10)$$

In equation (5.1.10) $\phi(x)$ and $\theta(t)$ are arbitrary functions and C , ξ_1 arbitrary constants, and $\Phi'(x) = \frac{1}{\beta(x)}, \theta'(t) = \frac{1}{\alpha(t)}$.

It may be mentioned here that the solution (5.1.10) to Liouville equation is completley new and does not seem to have been reported in the litrature. Further, choosing, $\theta(t) = -\log(1+\lambda_2)^{\frac{r}{2}}, \Phi(x) = \log(x+\lambda_1$ and $\xi_1 = \log\, c^{-\frac{1}{2}}$, where λ_1, λ_2 and r arbitrary constants, we get

$$u(x,t) = \frac{1}{n}\log\left[\frac{r}{2n}\frac{1}{(x+\lambda_1)(t+\lambda_2)}\sec h^2[\log\frac{(t+\lambda_2)^{\frac{r}{2}}}{(c(x+\lambda_1))^{\frac{1}{2}}}]\right]. \qquad (5.1.11)$$

Equation (5.1.11) represents an exact solution of Liouville equation reported by Bhutani et al (1992) [12] obtained via the isovector approach.

5.2 phi-four equation: $f(u) = u^3 - u$.

Case II (table 1, row 2). Corresponding to this case the equation (3.1) is reduced to the following ordinary differential equation.

$$\frac{b_1}{a_1}\eta'' + \eta^3 - \eta = 0. \qquad (5.2.1)$$

On solving equation (5.2.1), we arrive at the following form of

$$\eta = \sqrt{2}\,Sech\left(-\sqrt{\frac{a_1}{b_1}}(\xi+\xi_2)\right), \qquad (5.2.2)$$

where ξ_2 is a constant of integration.

In combining the equation (3.1) and (5.2.2) for the present case, we obtain

$$u = \sqrt{2}\ \sec h\left[-\sqrt{\frac{a_1}{b_1}}(x-\frac{b_1}{a_1}t)+\xi_2\right]. \qquad (5.2.3)$$

Equation (5.2.3) represents a solition type solution and has an importance of its own.

For the clear and quick insight into the results, the ordinary differential equations and the solutions / reduced forms are presented in tabular forms (table 2) for the last four cases in the table (1). Some of the results obtained here are totally new whereas some of them are known in the literature [13].

Table (1)

No. of case	f(u)	The infinitesimals			The invariants	
		A	B	C	ξ	η
1	Liouville equation $e^{nu}, n \neq 0$ [5]	$\alpha(t)$	$\beta(x)$	$\frac{1}{n}(\alpha'(t)+\beta'(x))$	$\Phi(x)-\theta(t)$	$\eta(\xi)=\frac{e^{-nu}}{\alpha(t)\beta(x)}$
2	Phi-four equation u^3-u [6]	a_1	b_1	0	$x-\frac{b_1}{a_1}t$	$\eta(\xi)=u$
3	Sine-Gordon equation $k_o \sin u$ [8]	a_2	b_2	0	$x-\frac{b_2}{a_2}t$	$\eta(\xi)=u$
4	klein-Gordon equation $c_1e^{\beta_o u}+c_2e^{2\beta_o u}+c_3e^{-\beta_o u}+c_4e^{-2\beta_o u}$ [11]	a_3	b_3	0	$x-\frac{b_3}{a_3}t$	$\eta(\xi)=u$
5	Mikhailov equation $c_2e^{2\beta_o u}+c_3e^{-\beta_o u}$ [11]	a_4	b_4	0	$x-\frac{b_4}{a_4}t$	$\eta(\xi)=u$
6	Double-Sine Gordon equation $c_1\sin\beta_o u+c_2\sin 2\beta_o u$ [11]	a_5	b_5	0	$x-\frac{b_5}{a_5}t$	$\eta(\xi)=u$

where $\Phi(x)=\int\frac{dx}{\beta(x)}$ and $\theta(t)=\int\frac{dt}{\alpha(t)}$ and a_i, b_i, i = 1,2,3,4 and 5, c_j, j=1,2,3 and 4 are arbitrary constants.

Table (2)

Equation	Ordinary Differential Equations	Solutions / Reduced Forms
(5.3) Sine-Gordon Equation	$-\frac{b_2}{a_2}\eta'' + k_0 \sin\eta = 0$ (5.3.1) if $w = e^{i\eta}$ Equation (5.3.1) gets transformed to $(ww'' - w'^2) + \frac{k_0 a_2}{2b_2}(w^3 - w) = 0$ (5.3.2)	(1) $w'^2 = \frac{-k_0 a_2}{b_2}(w^3 + w) + cw^2$ (5.3.3) where C is a constant of integration. This equation is solvable in terms of the elliptic Jacobian functions. So is of Painleve type [15]. 2-In the Case - $a_2 k_0 = b_2, c = 2$. We obtain the exact solution $u(x,t) = -2i \log\left[\tan\frac{1}{2}\left(x + k_0 t + \xi_0\right)\right]$ (5.3.4) where ξ_0 a constant of integration. this solution is completely new.
(5.4) General form Of klein - Gordon equation	$\frac{-b_3}{a_3}\eta'' = c_1 e^{\beta_0\eta} + c_2 e^{2\beta_0\eta} + c_3 e^{-\beta_0\eta} + c_4 e^{-2\beta_0\eta}$ (5.4.1) if $w = e^{i\eta}$ Equation (5.4.1) gets transformed to : $ww'' - w'^2 = \frac{-a_3\beta_0}{b_3}(c_1 w^3 + c_2 w^4 + c_3 w + c_4)$. (5.4.2)	$w'^2 = \frac{-a_3\beta_0}{b_3}(2c_1 w^3 + c_2 w^4 - 2c_3 w - c_4) + cw^2$ (5.4.3) Where c is a constant of integration. The solution of equation (5.4.3) can be expressed in terms of elliptic Jacobian function and so is of Painleve type .
(5.5) Mikhailov equation	$\frac{-b_4}{a_4}\eta'' = c_2 e^{2\beta_0\eta} + c_3 e^{-\beta_0\eta}$ (5.5.1) if $w = e^{i\eta}$ Equation (5.5.1) gets transformed to : $ww'' - w'^2 = \frac{-a_4\beta_0}{b_4}(c_2 w^4 + c_3 w)$ (5.5.2)	(1) $w'^2 = \frac{-a_4\beta_0}{b_4}(c_2 w^4 - 2c_2 w) + cw^2$ (5.5.3) (2) if $b_4 = -a_4\beta_0, c = -3c_2, c_3 = -c_2$ and using the transformation $\theta^2(\xi) = \frac{w(\xi)}{w(\xi) + 2}$. (5.5.4) then we get : $\theta'(\xi) = \frac{\sqrt{c_2}}{2}(3\theta^2 - 1)$

		when integrated yields : $\theta(\xi) = \mp\frac{1}{\sqrt{3}}\tanh\left(\frac{\sqrt{3c_2}}{2}(\xi+\xi_o)\right)$ (5.5.5) Then we have the following new exact solution of Mikhailov equation . $U(x,t) = \log\left[\frac{2-2\operatorname{sech}^2\left(\frac{\sqrt{3c_2}}{2}(x+\beta_o t+k)\right)}{2+2\operatorname{sech}^2\left(\frac{\sqrt{3c_2}}{2}(x+\beta_o t+k)\right)}\right]$ where k is an arbitrary constant.
(5.6) Double Sine-Gordon equation	$\frac{-b_5}{a_5}\eta'' = c_1\sin\beta_o\eta + c_2\sin 2\beta_o\eta$ (5.6.1) if $w = e^{i\beta_o\eta}$ then the equation (5.6.1) gets transformed to : $ww'' - w'^2 =$ $\frac{-a_5\beta_o}{2b_5}\left[c_1(w^3-w)+c_2(w^4-1)\right].$ (5.6.2)	(1) $W'^2 = \frac{-a_5\beta_o}{b_5}\left(c_1(w^3+w)+\frac{c_2}{2}(w^4+1)\right)+cw^2$ (5.6.3) where c is an arbitrary constant of integration . This equation is solvable in terms of the elliptic- Jacobian functions and so is of *Painleve type* . (2) for the choices . $-a_5\beta_o = b_5, c_1 = c_2 = 2$, and c=-6 , then using the transformation $\theta^2(\xi) = \frac{w(\xi)+2-\sqrt{3}}{w(\xi)+2+\sqrt{3}}$. (5.6.4) We get the ODE . $2\theta'(\xi) = \pm(3+\sqrt{3})\left(\theta^2(\xi) - \frac{(3-\sqrt{3})^2}{6}\right)$ (5.6.5) The solution of (5.6.5) obtained as : $\theta(\xi) = \mp\frac{(3-\sqrt{3})}{\sqrt{6}}\tanh\frac{\sqrt{6}}{2}(\xi+\xi_o)$ Where ξ_0 is a constant of integration . Therefore a new exact solution to the Double Sine-Gordon is given as : $u(x,t) = -\frac{i}{\beta_o}\log\left[\frac{(\sqrt{3}-1)-\operatorname{sech}^2\frac{\sqrt{6}}{2}(x+\beta_o t+k_1)}{(\sqrt{3}-1)+(2-\sqrt{3})\operatorname{sech}^2\frac{\sqrt{6}}{2}(x+\beta_o t+k_1)}\right]$ Where k_1 is an arbitrary constant.

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Appendix [A]

Substituting the values of different derivatives of S(u) in equation (4.2) and collecting the coefficients of various powers of u_t, u_x, u_t, u_x, ..., etc;we get.

$$F(L, u, S(u)) = (A_{xt}+C_{\psi u}-Af^{\backslash}(u)\, u_t+(C_{u\psi}+B_{xt}-Bf^{\backslash}(u))u_x$$
$$+ A_{xu}u_x^2+(B_{xu}+C_{uu}+A_{ut})\, u_x u_t+A_{uu}u_x u_t^2$$
$$+ B_{uu}u_t u_x^2+B_t u_{xx}+(A_t+B_x+C_u)u_{tx}+A_x u_{tt}$$
$$+ A_u u_x u_{tt}+2A_u u_t u_{tx}+B_u u_t u_{\psi x}+2B_u u_x u_{tx}$$
$$+ Au_{tt\psi}+Bu_{txx}+C_{xt}-f^{\backslash}(u)C.$$

Using the equation (3.1) in (a1) and replacing utx byy f(u), u_{ttx} by $f^{\backslash}(u)ut$ and u_{txx} by $f^{\backslash}(u)u_x$, we get:

$$F(L, u,S(u)) = [C_{ut}+B_{xt}+2B_u f(u)]u_x$$
$$+[A_{xt}+C_{xu}+2A_u f(u)]u_t$$
$$+[B_{xu}+A_{ut}+C_{uu}]u_x u_t$$
$$+B_{xx}u^2{}_x+A_{xu}u^2{}_t+A_{uu}u_x u^2{}_t$$
$$+B_{uu}u_t u^2{}_x+B_t u_{xx}+A_x u_{tt}$$
$$+A_u u_\psi u_{tt}+B_u u_t u_{xx}+C_{xt}-f^{\backslash}(u)C(A_t+B_x+C_u f(u)\text{-(A2)}.$$

"الحلول التماثلية للمعادلات الزائدية نصف الخطية $U_{xt} = F(u)$ بواسطة الطريقة المتماثلة"

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المعادلة الزائدية نصف الخطية ذات الأهمية الفيزيائية والرياضية $U_{xt} = F(u)$ حيث $F(u)$ داله إختيارية قد تم حلها بواسطة الطريقة المتماثلة لاستبزج وذلك فى حالات معادله ليوفيل، معادله فاى فور، معادلة ساين جوردن، معادلة كلين جوردن، معادلة ميخالوف، معادلة تضاعف ساين جوردن.

التقريبات الصغيرة والمتغيرات المتماثلة والمتغيرات التابعة والتحويلات الى دوال قابلة للتكامل أو حلول تامة أو مضبوطة جدولتها للدالة $F(u)$ في الحالات الفيزيائية المشار إليها ويجدر الاشارة إلى أن قد تم الحصول على حلول مضبوطة وجديدة لم تظهر من قبل لبعض حالات الدالة $F(u)$.