PLANE STRESS ANALYSIS OF SIMPLY SUPPORTED RECTANGULAR PLATES BY THE NODAL LINE FINITE DIFFERENCE METHOD

تحليل الألواح المستطيلة بسيطة الارتكاز لحالة الاجهادات المستوية

بطريقة الفروق المحددة للخطوط

BY

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الخلاصة – يتناول هذا البحث تحليل الألواح المستطيلة بسيطة الارتكاز لحالة الاجهادات المستوية وذلك باستخدام طريقة الفروق المحددة للخطوط nodal line finite difference method التي ابتكرها الباحث من قبل وطبقها في تحليل الألواح المعرضة لقوى عمودية على مستواها . ولقد تم استخدام دوال مثلثية أساسية في البحث الحالي وذلك للتعبير عن مركبتي الازاحة بحيث تفي هذه الدوال مسبقا بشرط الارتكاز البسيط عند حافتين متقابلتين من حواف اللوح . بذلك أمكن تحويل المعادلتين التفاضليتين الجزئيتين الآنيتين اللتان تحكمان سلوك هذا النوع من الألواح الى معادلتي الفروق الآنية باستخدام الفروق المحددة . ولقد تم حل بعض الأمثلة لنماذج من الألواح المستطيلة لحالة الاجهادات المستوية باستخدام طريقة الباحث وتم عمل مقارنة للنتائج التي تم الحصول عليها وأظهرت المقارنة كفاءة الطريقة وامكانية استخدامها لحل مشاكل أخرى في هذا المجال .

ABSTRACT: *The nodal line finite difference method developed earlier by the Author has been successfully applied to bending analysis of rectangular plates. In the present work, the method is extended to solve plane stress problems of rectangular plates. The analysis deals with the in-plane displacements and requires the solution of two simultaneous second-order partial differential equations. These equations are transformed into two simultaneous ordinary differential equations which in turns are cast in two simultaneous nodal line difference equations. Numerical results obtained by the present technique have shown good agreement with those obtained by the finite strip method and this indicates the applicability and the power of the present technique.*

INTRODUCTION

The nodal line finite difference method NLFDM is a semi-analytical approach developed recently by the Author. This method has been very promising in reducing the computational efforts and core requirements for a class of two and three dimensional stress analysis. The description formulation of the method are oriented particularly towards the analysi plates and shell structures. The method based on reducing the p differential equations, which describe the equilibrium conditions structure, to ordinary differantial equations by adopting continuou functions satisfying a priori the boundary conditions in one d These ordinary differential equations are then transformed into a difference equations by means of the finite difference technique. is similar to the finite strip method FSM developed by CHUNG (1,' both of them call for the use of basic functions at nodal lines line finite difference method NLFDM was first developed by th for bending analysis of simply supported rectangular plates a Author (7,8,9) for other plate bending problems. These ap proven the validity and the power of the method and possibility of extending the method to the handling of st other two and three dimensional problems.

The purpose of the present work is to develop a nodal line finite difference solution for plane stress analysis of rectangular plates. The analysis and design of such plates can be encountered in a number of important structures, such as butteresses, deep beams and shear walls. In such type of structures there exists only in-plane displacement which can be resolved into two components parallel to the rectangular coordinate axes. Accordingly, the equilibrium and the compatibility conditions of such plates are described by two simultaneous partial differential equations. In the application of the nodal line finite difference method for plane stress analysis of rectangular plates, the plate is divided into a mesh of fictitious nodal lines parallel to one of the coordinate axes. Two basic functions are chosen to express the displacement variation along these nodal lines with the stipulation that such basic functions should satisfy a priori the boundary conditions at the ends of the nodal lines. The partial differential equations are then transformed into two nodal line simultaneous difference equations by using the finite difference technique. In the present work, thin elastic isotropic rectangular plates with two opposite simply supported ends are analyzed. The results obtained are in close agreement with those of the same conditions worked out by CHUNG (3).

METHOD OF ANALYSIS

a) Nodal Line Difference Equations

In plane stress analysis of plates, only deformations in the plane of the plate due to in-plane forces are considered. These deformations can be expressed in terms of two components u and v parallel to the rectangular coordinate axes x and y. Detailed derivation of the differential equations relating the displacement components u and v with external load components Px and Py is quite simple and can be found in many references. For isotropic thin elastic rectangular plates, these differential equations are

$$\left.\begin{aligned} 2\,u^{\cdot\cdot} + (1-\nu)\,u'' + (1+\nu)\,v^{\cdot\prime} &= -\frac{2}{D}\,P_x \\ (1-\nu)\,v^{\cdot\cdot} + 2\,v'' + (1+\nu)\,u^{\cdot\prime} &= -\frac{2}{D}\,P_y \end{aligned}\right\} \qquad (1)$$

where $(\ )^{\cdot} = \frac{\partial}{\partial x}$, $(\ )' = \frac{\partial}{\partial y}$.

ν = Poisson's ratio and

$D = \frac{Et}{(1-\nu^2)}$ is the in-plane stiffness

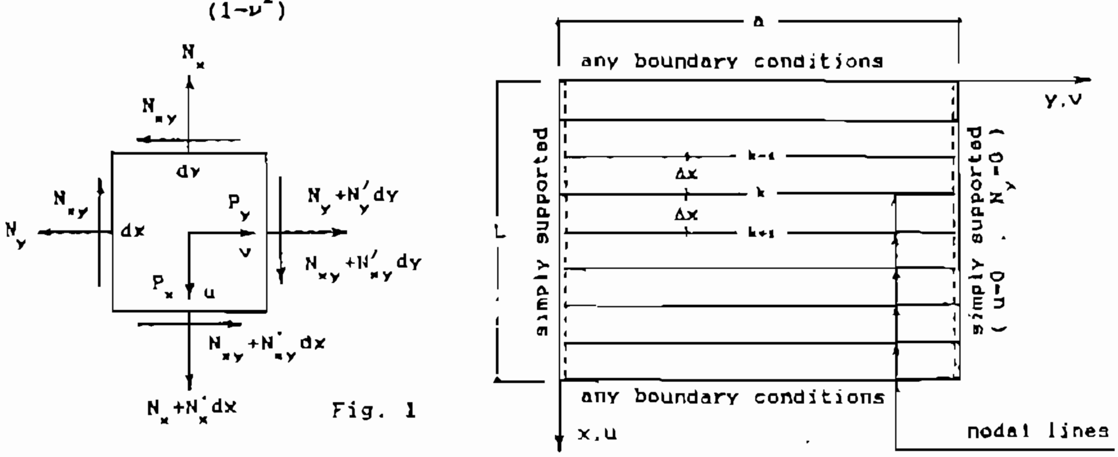


Fig. 1

The solution of plane stress problems by the nodal line finite difference method requires the division of the plate into a mesh of parallel fictitious nodal lines in one direction as shown in Fig. 1. The displacement functions at each nodal line are expressed as summation of the terms of the basic functions fitting the boundary conditions at the ends of the nodal lines multiplied by nodal line parameters. These parameters are assumed as single variable functions in the direction perpendicular to the nodal lines. For rectangular plates simply supported at two opposite ends, the displacement functions at any nodal line k are proposed as

$$\left.\begin{aligned} u_k &= \sum_{m=1}^{r} U_{m,k}(x) \sin \frac{m\pi}{a} y = \sum_{m=1}^{r} U_{m,k} \sin k_m y \\ v_k &= \sum_{m=1}^{r} V_{m,k}(x) \cos \frac{m\pi}{a} y = \sum_{m=1}^{r} V_{m,k} \cos k_m y \end{aligned}\right\} \qquad (2)$$

The adopted basic functions have uncoupling property and therefore each term of the series can be analyzed separately.
Substitution of equation (2) into equation (1) yields.

$$\left.\begin{aligned} \sum_{m=1}^{r} \left[2 U^{\cdot\cdot}_{m,k} - (1-\nu) k_m^2 U_{m,k} - (1+\nu) k_m V^{\cdot}_{m,k} \right] \sin k_m y &= -\frac{2}{D} P_{x,k} \\ \sum_{m=1}^{r} \left[(1-\nu) V^{\cdot\cdot}_{m,k} - 2 k_m^2 V_{m,k} + (1+\nu) k_m U^{\cdot}_{m,k} \right] \cos k_m y &= -\frac{2}{D} P_{y,k} \end{aligned}\right\} \qquad (3)$$

The applied loads must also be resolved into series similar to the displacement functions as follows

$$\left.\begin{aligned} P_{x,k} &= \sum_{m=1}^{r} q^{x}_{m,k} \sin k_m y \\ P_{y,k} &= \sum_{m=1}^{r} q^{y}_{m,k} \cos k_m y \end{aligned}\right\} \qquad (4)$$

Upon substitution of equation (4) into equation (3), we obtein the following relationships for each term of the basic functions

$$\left.\begin{aligned} \left[2 U^{\cdot\cdot}_{m,k} - (1-\nu) k_m^2 U_{m,k} - (1+\nu) k_m V^{\cdot}_{m,k} \right] &= -\frac{2}{D} q^{x}_{m,k} \\ \left[(1-\nu) V^{\cdot\cdot}_{m,k} - 2 k_m^2 V_{m,k} + (1+\nu) k_m U^{\cdot}_{m,k} \right] &= -\frac{2}{D} q^{y}_{m,k} \end{aligned}\right\} \qquad (5)$$

By applying the central finite difference technique, we get

$$\left.\begin{aligned} \left[1 \quad C^1_m \quad -C^3_m \quad 0 \quad 1 \quad -C^1_m \right] \{\delta_m\} &= -\frac{a^2}{D\lambda^2} q^{x}_{m,k} \\ \left[-C^1_m \quad C^2_m \quad 0 \quad -C^4_m \quad C^1_m \quad C^2_m \right] \{\delta_m\} &= -\frac{a^2}{D\lambda^2} q^{y}_{m,k} \end{aligned}\right\} \qquad (6)$$

where $\lambda = \frac{a}{\Delta x}$, $\psi_m = k_m \Delta x$, $C^1_m = \frac{1+\nu}{4} \psi_m$, $C^2_m = \frac{1-\nu}{2}$,

$C^3_m = (2 + C^2_m \psi^2_m)$, $C^4_m = \{(1-\nu) + \psi^2_m\}$ and

$\{\delta_m\} = \{ U_{m,k-1} V_{m,k-1} U_{m,k} \quad V_{m,k} \quad U_{m,k+1} V_{m,k+1} \}^T$

Equations (6) represent the two simultaneous nodal line difference equations required for the present plane stress analysis. Application of equation (6) at each nodal line of the plate yields uncoupled system of linear algebraic equations. The final matrix is a square band matrix having a small band width equal to 7.

b) Internal Forces.

For an elastic isotropic thin plate deformed by in-plane forces, the internal forces per unit length at any point are given by

$$\left.\begin{aligned} N_x &= D\,(u' + \nu\, v') \\ N_y &= D\,(v' + \nu\, u') \\ N_{xy} &= \frac{1-\nu}{2}\, D\,(u' + v') \end{aligned}\right\} \qquad (7)$$

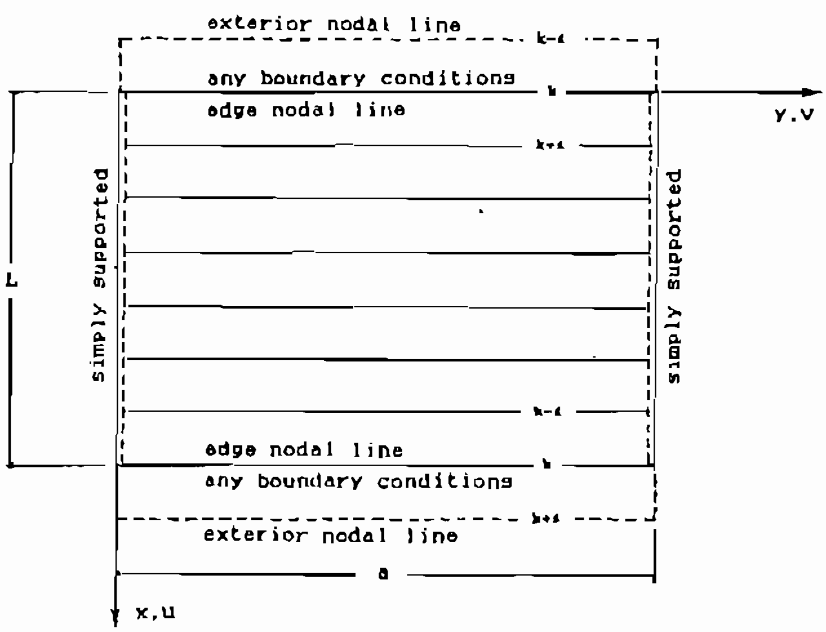
Once the nodal Line parameters $U_{m,k}$ and $V_{m,k}$ at each nodal line are obtained, it is simple matter to determine the internal forces at any point of the plate. By applying the central finite difference technique in the direction perpendicular to the nodal lines, the internal forces at any nodal lin k can be written as

$$\left.\begin{aligned} N_{x,k} &= \frac{D\lambda}{2a} \sum_{m=1}^{r} \sin k_m y \,[-1 \quad 0 \quad 0 \quad -2\nu\psi_m \quad 1 \quad 0]\,\{\delta_m\} \\ N_{y,k} &= \frac{D\lambda}{2a} \sum_{m=1}^{r} \sin k_m y \,[-\nu \quad 0 \quad 0 \quad -2\,\psi_m \quad \nu \quad 0]\,\{\delta_m\} \\ N_{xy,k} &= \frac{1-\nu}{2}\,\frac{D\lambda}{2a} \sum_{m=1}^{r} \cos k_m y \,[\,0 \quad -1 \quad 2\psi_m \quad 0 \quad 0 \quad 1\,]\,\{\delta_m\} \end{aligned}\right\} \qquad (8)$$

c) Boundary Conditions

The boundary conditions are the conditions at the edges which must be prescribed in advance in order to obtain the solution of the specific plate problem. The adopted basic functions should satisfy a priori the boundary conditions at the two opposite ends perpendicular to the nodal lines. The other two opposite edges can take any combination of boundary conditions.

Fig. 2



Plane stress analysis of rectangular plates by the proposed technique requires the application of the nodal line difference equations (6) at each nodal line of the plate including the edge nodal lines. Each edge nodal line difference equation will introduce one additional exterior nodal line.

According to the prescribed boundary conditions, the exterior nodal line parameters $U_{m,k-1}$, $V_{m,k-1}$, $U_{m,k+1}$ and $V_{m,k+1}$ have to be expressed in terms of the edge and adjacent interior nodal line parameters. The exterior nodal line parameters are connected to those of the edge and adjacent interior nodal line parameters for each terms of the basic functions through the following relations.

1- Simply supported edge $\{ v = 0 \quad , \quad N_x = 0 \}_k$

$$\left.\begin{aligned} U_{m,k-1} &= U_{m,k+1} \\ V_{m,k-1} &= -V_{m,k+1} \\ U_{m,k+1} &= U_{m,k-1} \\ V_{m,k+1} &= -V_{m,k-1} \end{aligned}\right\} \tag{9}$$

2- Clamped edge $\{ u = 0 \quad , \quad v = 0 \}_k$

$$\left.\begin{aligned} U_{m,k-1} &= -U_{m,k+1} \\ V_{m,k-1} &= -V_{m,k+1} \\ U_{m,k+1} &= -U_{m,k-1} \\ V_{m,k+1} &= -V_{m,k-1} \end{aligned}\right\} \tag{10}$$

3- Free edge $\{ N_x = 0 \quad , \quad N_{xy} = 0 \}_k$

$$\left.\begin{aligned} U_{m,k-1} &= -2\nu\psi_m V_{m,k} + U_{m,k+1} \\ V_{m,k-1} &= 2\psi_m U_{m,k} + V_{m,k+1} \\ U_{m,k+1} &= 2\nu\psi_m V_{m,k} + U_{m,k-1} \\ V_{m,k+1} &= -2\psi_m U_{m,k} + V_{m,k-1} \end{aligned}\right\} \tag{11}$$

4- Loaded edge $\{ N_x = P(y) \; . \; N_{xy} = q(y) \}_k$

$$\left\{ N_x = P(y) = \sum_{m=1}^{r} P_m \sin k_m y \;,\; N_{xy} = q(y) = \sum_{m=1}^{r} q_m \cos k_m y \right\}_k$$

$$\left.\begin{aligned} U_{m,k-1} &= -2\nu\psi_m V_{m,k} + U_{m,k+1} - \frac{2a}{D\lambda} P_m \\ V_{m,k-1} &= 2\psi_m U_{m,k} + V_{m,k+1} - \frac{2a}{D\lambda}\frac{2}{1-\nu} q_m \\ U_{m,k+1} &= 2\nu\psi_m V_{m,k} + U_{m,k-1} + \frac{2a}{D\lambda} P_m \\ V_{m,k+1} &= -2\psi_m U_{m,k} + V_{m,k-1} + \frac{2a}{D\lambda}\frac{2}{1-\nu} q_m \end{aligned}\right\} \tag{12}$$

NUMERICAL EXAMPLES

In order to demonstrate the validity and the accuracy of the proposed solution technique, a number of rectangular plate problems has been analyzed. The present analysis is restricted to thin isotropic rectangular plates having two opposite simply supported edges. Due to symmetry, only odd terms of the basic function used to describe the displacement u as well as the even terms of the basic function used to describe the displacement v are considered.

EXAMPLE 1 : To check the accuracy of the proposed technique, a square plate subjected to uniformly distributed line load acting at its top edge has been analyzed. In this analysis, the plate is divided into a mesh of fictitious nodal lines with $\Delta x = L/24$. The analysis has been carried out for nine terms of the used basic functions. The results obtained are presented in Fig. 3. For the purpose of comparison, a copy of the results obtained by CHEUNG [3] from the finite strip solution is provided in APPENDIX I. Due to the absence of the value of poisson's ratio, ν, in CHEUNG's solution, a value of $\nu=1/6$ is considered in the present analysis. The comparison has shown a good agreement.

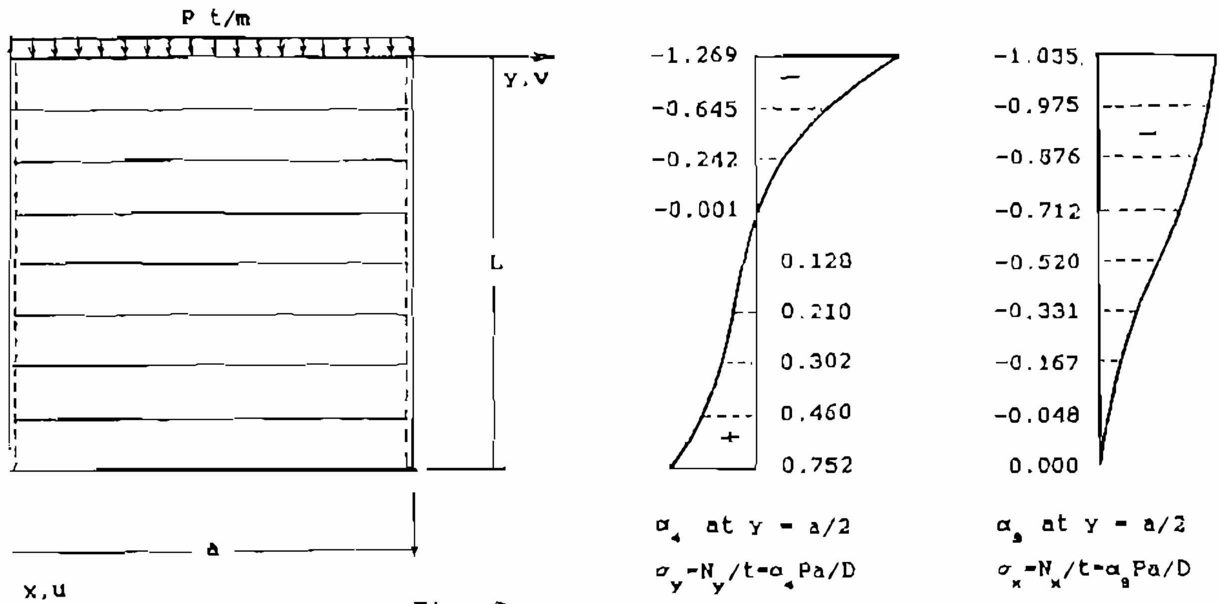


Fig. 3

:XAMPLE 2 : This example deals with studying the)ffect of the load position on the distribution of he displacements and the internal forces. A .quare plate subjected to uniform distributed load .cting at any nodal line within the plate .ncluding the top and the bottom edge nodal lines)as been analyzed, Fig. 4. Information regarding :he distance between the nodal lines, Δx, the 1umber of terms of the basic function used and the)oisson's ratio, ν, are taken as those mentioned in the previous example. The numerical values of the obtained results are presented in Tables 1, 2, 3, 4 and 5. Furthermore, these numerical values 1re plotted in the curves given in Fig. 5.

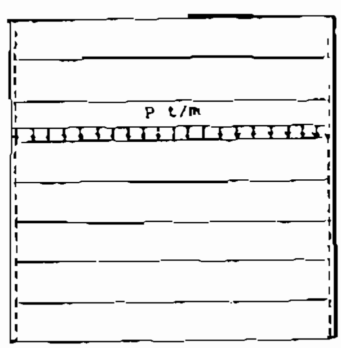


Fig. 4

Table 1. Coefficients a_1 at y=a/2 for the displacement component u $u = a_1 Pa/D$

x/L	Distributed line load position x/L								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.000	0.863	0.749	0.639	0.539	0.458	0.400	0.363	0.342	0.320
0.125	0.751	0.760	0.658	0.558	0.476	0.417	0.379	0.357	0.343
0.250	0.640	0.658	0.677	0.580	0.499	0.440	0.401	0.379	0.364
0.375	0.540	0.550	0.579	0.613	0.536	0.470	0.440	0.416	0.401
0.500	0.459	0.476	0.499	0.538	0.590	0.536	0.499	0.476	0.459
0.625	0.401	0.416	0.440	0.478	0.536	0.613	0.579	0.558	0.540
0.750	0.364	0.379	0.401	0.440	0.499	0.580	0.677	0.658	0.640
0.875	0.343	0.357	0.379	0.417	0.476	0.558	0.658	0.760	0.751
1.000	0.328	0.342	0.363	0.400	0.458	0.539	0.639	0.749	0.863

Table 2. Coefficients a_2 at y=0.0 for the displacement component v $v = a_2 Pa/D$

x/L	Distributed line load position x/L								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.000	0.480	0.475	0.424	0.372	0.327	0.292	0.270	0.256	0.247
0.125	0.131	0.179	0.211	0.205	0.190	0.176	0.165	0.158	0.154
0.250	0.000	0.017	0.064	0.108	0.117	0.117	0.114	0.111	0.109
0.375	-0.056	-0.049	-0.033	0.018	0.069	0.084	0.089	0.090	0.090
0.500	-0.078	-0.076	-0.069	-0.053	0.000	0.053	0.069	0.076	0.078
0.625	-0.090	-0.090	-0.089	-0.084	-0.069	-0.018	0.033	0.049	0.056
0.750	-0.109	-0.111	-0.114	-0.117	-0.117	-0.108	-0.064	-0.017	0.000
0.875	-0.154	-0.158	-0.165	-0.176	-0.190	-0.205	-0.211	-0.179	-0.131
1.000	-0.247	-0.256	-0.270	-0.292	-0.327	-0.372	-0.424	-0.475	-0.480

Table 3. Coefficients a_3 at y=a/2 for the normal force N_x $N_x = \sigma_x t = a_3 P$

x/L	Distributed line load position x/L								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.000	-1.035	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.125	-0.975	-0.474	0.036	0.047	0.052	0.052	0.051	0.049	0.048
0.250	-0.876	-0.800	-0.363	0.163	0.178	0.180	0.177	0.172	0.167
0.375	-0.712	-0.722	-0.713	-0.108	0.336	0.346	0.346	0.340	0.331
0.500	-0.520	-0.531	-0.533	-0.572	0.000	0.522	0.533	0.531	0.520
0.625	-0.331	-0.340	-0.346	-0.346	-0.336	0.108	0.713	0.722	0.712
0.750	-0.167	-0.172	-0.177	-0.180	-0.178	-0.163	0.363	0.880	0.876
0.875	-0.048	-0.049	-0.051	-0.052	-0.052	-0.047	-0.036	0.474	0.975
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.035

Table 4. Coefficients a_4 at y=a/2 for the normal force N_y $N_y = \sigma_y t = a_4 P$

x/L	Distributed line load position x/L								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.000	-1.269	-1.116	-1.120	-1.057	-0.965	-0.000	-0.818	-0.780	-0.752
0.125	-0.645	-0.628	-0.581	-0.560	-0.554	-0.521	-0.493	-0.474	-0.460
0.250	-0.242	-0.291	-0.271	-0.250	-0.291	-0.308	-0.310	-0.307	-0.302
0.375	-0.001	-0.031	-0.080	-0.090	-0.095	-0.159	-0.193	-0.207	-0.210
0.500	0.128	0.114	0.079	0.012	0.000	-0.012	-0.079	-0.114	-0.128
0.625	0.210	0.207	0.193	0.159	0.095	0.090	0.088	0.031	0.001
0.750	0.302	0.307	0.310	0.308	0.291	0.250	0.271	0.291	0.242
0.875	0.460	0.474	0.493	0.571	0.554	0.500	0.581	0.628	0.645
1.000	0.752	0.780	0.810	0.880	0.965	1.057	1.120	1.116	1.269

Table 5. Coefficients a_5 at y=0.0 for the shearing force N_{xy} $N_{xy} = \tau t = a_5 P$

x/L	Distributed line load position x/L								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.175	0.727	0.022	0.498	0.382	0.318	0.279	0.255	0.242	0.234
0.250	0.607	0.705	0.915	0.622	0.513	0.452	0.415	0.395	0.382
0.375	0.630	0.653	0.700	0.944	0.669	0.575	0.527	0.503	0.480
0.500	0.571	0.506	0.610	0.693	0.950	0.693	0.618	0.586	0.571
0.625	0.400	0.503	0.527	0.575	0.669	0.944	0.708	0.653	0.630
0.750	0.382	0.395	0.415	0.452	0.513	0.622	0.915	0.705	0.607
0.075	0.234	0.242	0.255	0.279	0.318	0.382	0.498	0.822	0.727
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

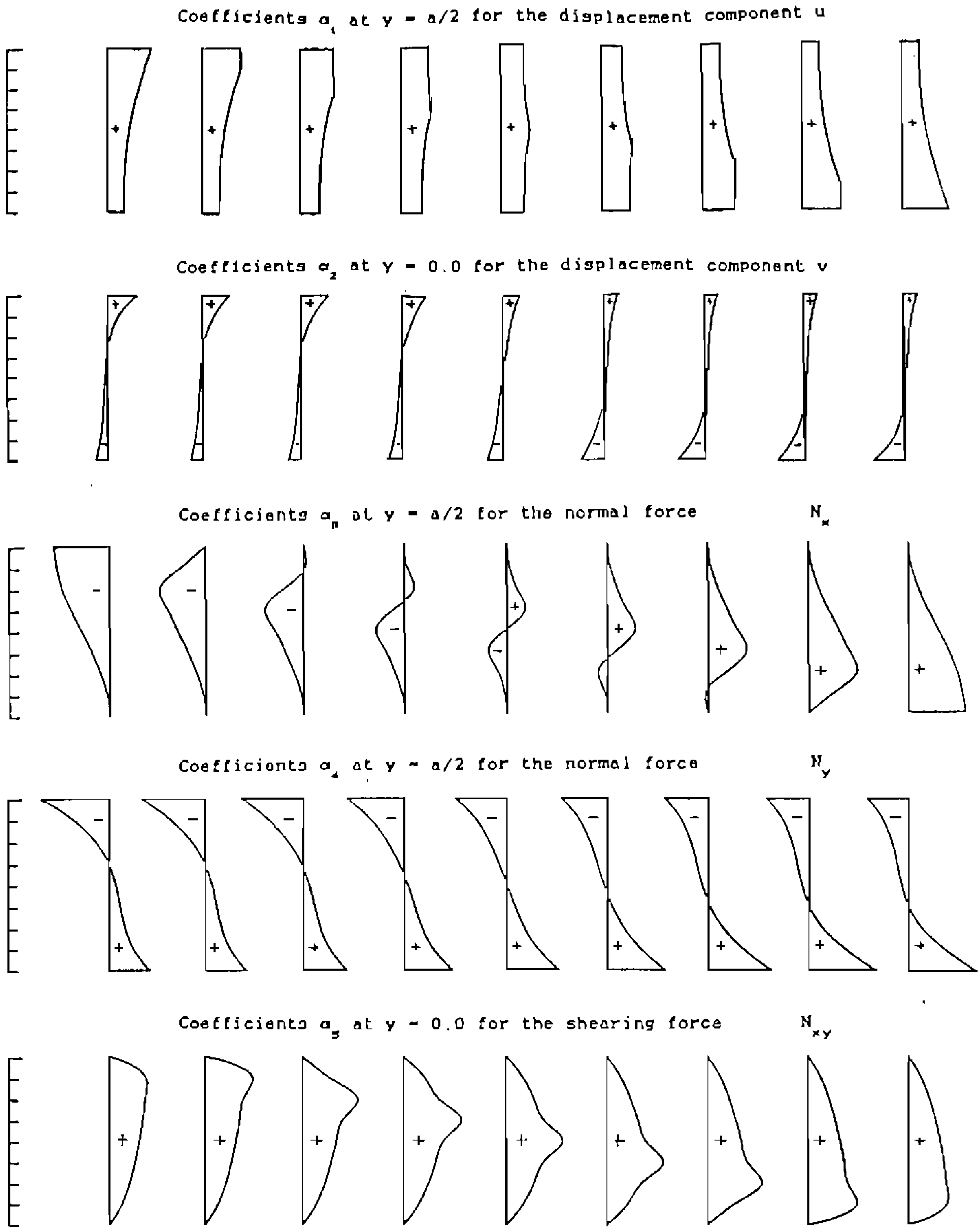

Coefficients α_1 at y = a/2 for the displacement component u
Coefficients α_2 at y = 0.0 for the displacement component v
Coefficients α_3 at y = a/2 for the normal force N_x
Coefficients α_4 at y = a/2 for the normal force N_y
Coefficients α_5 at y = 0.0 for the shearing force N_{xy}

Fig. 5

EXAMPLE 3 : This example deals with studying the aspect ratio effect on the distribution of the displacements and the internal forces. A set of rectangular plates with different ratios subjected to uniform distributed line load acting at the top edge has been analyzed. The analysis has been carried out for nine terms of the used basic functions. The distance between the nodal lines is taken as Δx = L/24 for aspect ratios 0.5 and 1.0, while Δx equal to L/48 for aspect ratios 1.5 and 2.0. The poisson's ratio, ν, is taken as that of the previous examples The obtained results are plotted in Figs. 6 and 7.

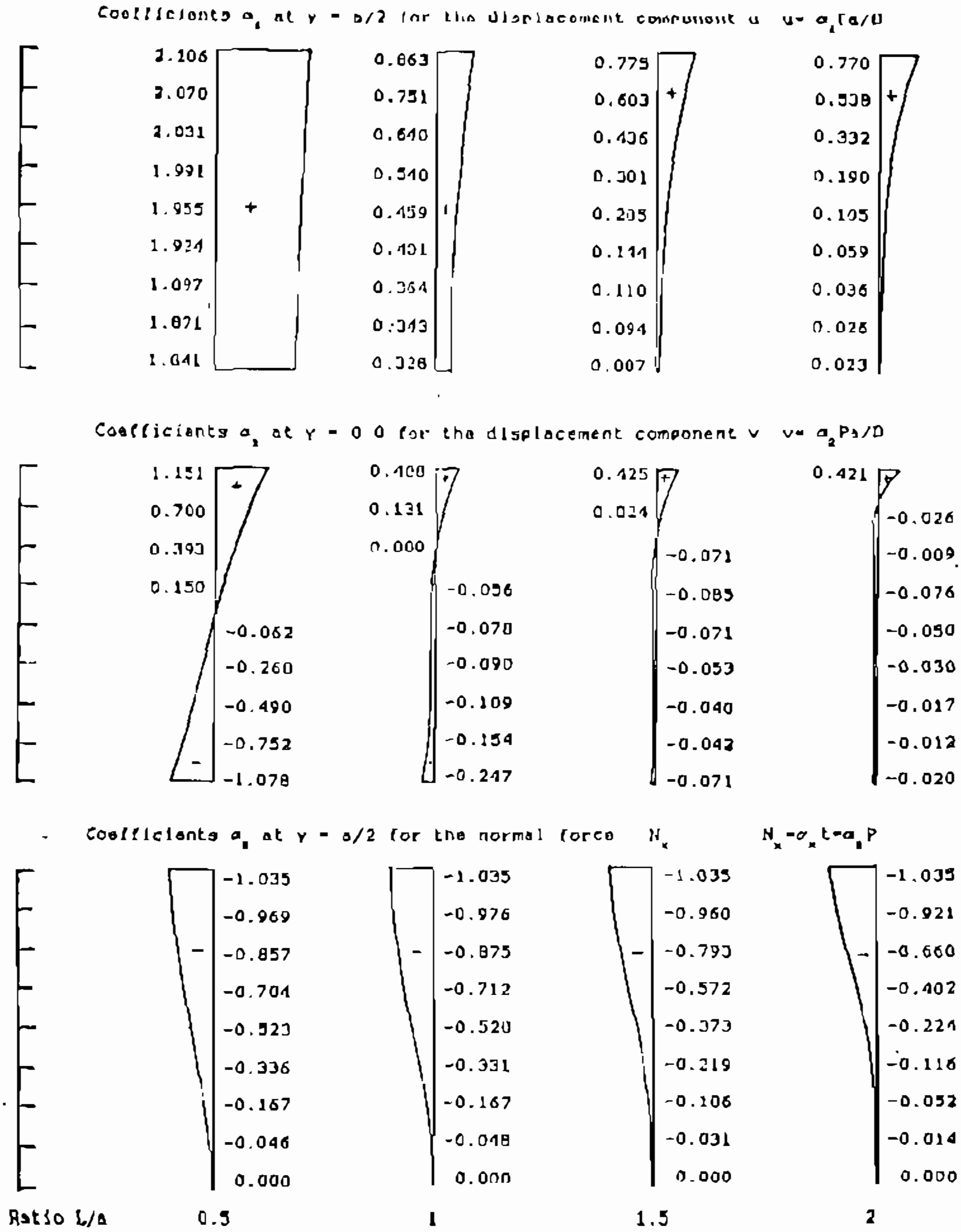


Fig. 6