

## ORDER MANIFOLD

By

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### INTRODUCTION

The connection of topology and ordering was introduced by L. Nachbin[2]. He gave most basic definition and proveds many generalization of well-known results concerning general topology and functional analysis. Since then, many authors have investigated the subject systematically.

In this paper we introduce the idea of order manifold manisold which is a generalization of differentiable manifold. We give some properties with examples.

### ORDER DIFFERENTIABLE MANIFOLDS

#### 1) definitions :

An ordered n-dimensional topological manifold  $M$  is an ordered space with a countable base for the topology which is locally order homeomorphic to  $\mathbb{R}^n$ .

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The last condition means that, for each point  $x \in M$ , there exists an open neighbourhood  $U$  of  $x$  and an order homeomorphism.

$$h : U \rightarrow U' \text{ onto an open set } U' \subseteq \mathbb{R}^n$$

**2) Examples :**

(i) Figure (1) shows two examples of order 1 - manifolds. It should be noticed that each order 1 - manifold has the totally order.

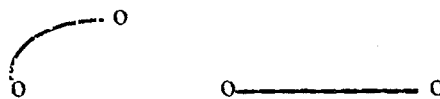


fig. 1

- (ii) The circle is not an order 1 - manifold although it is a 1 - manifold.
- (iii)  $\mathbb{R}^n$  with the usual topology and the vector order is an order  $n$  - manifold.
- (iv) Any open subset of an order  $n$  - manifold is again an order  $n$  - manifold.

If  $M$  is an order  $n$ -manifold and  $h : U \rightarrow U'$  is an order homeomorphism of an open subset  $U \subseteq M$  onto an open subset  $U' \subseteq \mathbb{R}^n$ , then  $h$  is called an order chart of  $M$  and  $U$  is the associated chart domain. The collection of order charts  $\{h_\alpha : U_\alpha \rightarrow U'_\alpha, \alpha \in \Omega\}$  is called an order atlas for  $M$  if.

$$\bigcup_{\alpha \in \Omega} U_\alpha = M$$

The first question to be answered is that : Given two order charts, both are order - homeomorphisms  $h_\alpha, h_\beta$  are defined on the intersection of their domain  $U_{\alpha\beta} := U_\alpha \cap U_\beta$ . Is  $h_{\alpha\beta}$  an order homeomorphism on which  $h_\beta \circ h_\alpha = h_{\alpha\beta}$ ?

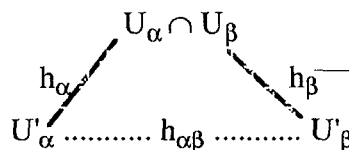


Fig. 2

Since the composition of two order homeomorphisms is again an order homeomorphism [3], it follows that the answer of the given question is yes.

**3) Theorem :** If  $M$  is an order  $m$ -manifold and  $N$  is an order  $n$ -manifold, then the product space  $M \times N$  is an order  $(m+n)$ -manifold.

**Proof :** The product of two ordered space is again an ordered space [3], and the dimension of the resulted space is the sum of dimensions of the two spaces. Also the  $U^n \times U^m$  is an order homeomorphism to  $U^{n+m}$ . Hence the rest of the proof is similar to the proof of the ordinary case.

As one expected an order atlas of an order manifold is order differentiable if all its order chart transformations are order differentiable. Also a continuous order mapping between two order differentiable manifolds defined similarly.

**4) Theorem :** Every order differentiable manifold,  $M$  possesses a countable order atlas.

**Proof :** Since the manifold  $M$  has a countable base, each open set  $U \in M$  contains an open set  $V \in \mathfrak{b}$  where  $\mathfrak{b}$  is a countable base for the order manifold. Thus one can select these order homeomorphisms from the differentiable atlas. This completes the proof.

The definition of order embedding was introduced in [2]. If  $R^k$  and  $R^n$  are given the usual topology and the vector order and  $k \leq n$ , then we conclude.

**5) Lemma :**  $R^k$  is an order embeded in  $R^n$ .

**Proof :**  $(x_1, \dots, x_k) - f \longrightarrow (x_1, \dots, x_k, 0, \dots, 0)$

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Is a continuous mapping from  $R^k$  to  $R^n$  and preseving order as well. Obviously  $f$  is one to one, hence the order embedding follows.

Using the previous lemma, one can easily proof the following.

**6) Theorem :** If  $M \neq \emptyset$  is an order  $n$  - manifold and  $k \leq n$ , then there exists and order embedding  $R^k \rightarrow M$ .

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