Mathematics

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Mathematics

Higher estimates for the solution of a heavy rigid body moving under the action of a Newtonian force field

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Abstract In this paper, we introduce a new solution of the Euler's dynamic equations for the rotational motion of a rigid bo dy about a fixed point under the action of a Newtonian force field. The components of the angular velocity vector for this solution are differing from the most famous cases. We assumed that the center of mass of the rigid body coincides with the fixed point and a restriction on an initial condition is applied. The obtained solution is represented graphically using most recent computer codes to describe the motion at any time and is considered as a modification of Euler's case.

Key words: *Euler's equations, Rigid body motion, Newtonian field*

Introduction:

The rotational motion of a rigid body about a fixed point in a Newtonian force field is one of the important problems in theoretical classical mechanics. This problem attracted the interest of many researchers during the last five decades e. g. [1-7]. The great importance of this research subject is due to the wide range of its applications in mechanics. To solve these problems we need to deal with intricate techniques because they are governed by a system contains six non-linear differential equations besides with three first integrals [8]. The exact solutions of such systems require an additional fourth algebraic first integral. Many researches realized such integral for famous special cases, which have some restrictions on the body center of mass location and on the torques acting on the body [9].

The perturbed rotational motion of a heavy solid close to regular precession with constant restoring moment was treated in [2] and [3]. The authors assumed some initial conditions to achieve the analytical solutions of the equations of motion using averaging method [10] up to the first and second approximations. The rotatory motion of a symmetric gyrostat about a fixed point when one component of the gyrostatic torque is applied and in the presence of some torques was considered in [4] and generalized in [5]. The motion of an electromagnetic gyroscope is investigated in [6] when a Newtonian field, perturbed moments and restoring ones are applied. The averaging technique [10] is used to obtain the first order approximate analytical solutions. The graphical representations of these solutions are presented to describe the motion at any instant. The rotational motion of that body under the action of a Newtonian force field with the application of the third component of a gyrostatic moment is investigated in [7]. The approximate periodic solutions of the governing

equations are obtained using the small parameter method of Poinacré [11]. This method and its modifications [12-13] are used in [14] to construct the periodic solutions of limiting case for the motion of a rigid body about a fixed point in a Newtonian force field.

The rotational motion of a heavy solid about a fixed point in the presence of a gyrostatic moment vector is presented in [15]. The authors supposed that the body has rapidly spinning about the major or the minor principal axis of the ellipsoid of inertia. Krylov-Bogoliubov-Mitropolski technique [10] is modified and used to achieve the periodic solutions of the equations of motion.

The perturbed self-excited rigid body problem with a fixed point is investigated in [16]. The averaging theory [17] is used to study the periodic orbits up to first order. In [18], the authors presented the possibility of constructing exact analytic solutions concerning the dynamic Euler equations of motion.

The spinning motion of the hovering magnetic top and its dynamic stability were analyzed in [19] and [20]. The numerical integration of a heavy magnetic top is investigated in [21].
Existence of periodic motions of a rigid body was investigated in [22]. The small parameter method was used to obtain the periodic solutions of the equations of motion. The center of mass of the body is slightly shifted from a dynamically symmetric axis. The generalization of this problem was treated in [23] when the body rotates under the action of a Newtonian field and in the presence of one component of the gyrostatic moment vector. A new exact solution of the equations of motion of a rigid body is investigated in [24] when the body moves under the action of a uniform force field. The author assumed that the center of mass of the body is located at meridional plane and the principal torques of inertia satisfied a simple algebraic condition.

In this work, we extend the previous studies when the rigid body moves under the action of a Newtonian force field arising from an attracting center located on the downward fixed axis. We assume that the center of mass of the body coincides with the fixed point (origin). The achieved solution is obtained after taking account some algebraic assumptions concerning on the moments of inertia. This solution is represented graphically, in the rest of this paper, to show the behavior of the body motion under the action of Newtonian force field. From this point of view,
the current study may be regarded as a modification of Euler's case for the motion of a rigid body.

2. Equations of motion

Consider the motion of a heavy rigid body that rotates about a fixed point O, in the body, under the influence of a Newtonian force field arising from an attracting center O_1 being located on a downward fixed axis passing through the fixed point O. Let $OXYZ$ be a fixed coordinate system and another moving one $Oxyz$ which is fixed in the body and whose axes are directed along the principal axes of inertia of the body with origin O. The equations of motion are given below [14]

$$A\dot{p}+(C-B)qr=Mg(\gamma_2 z_0-\gamma_3 y_0)+N(C-B)\gamma_2\gamma_3,$$
$$B\dot{q}+(A-C)rp=Mg(\gamma_3 x_0-\gamma_1 z_0)+N(A-C)\gamma_3\gamma_1,$$
$$C\dot{r}+(B-A)pq=Mg(\gamma_1 y_0-\gamma_2 x_0)+N(B-A)\gamma_1\gamma_2,$$
(1)

with

$$\dot{\gamma}_1=r\gamma_2-q\gamma_3,\quad \dot{\gamma}_2=p\gamma_3-r\gamma_1,$$
$$\dot{\gamma}_3=q\gamma_1-p\gamma_2,$$

(2)

where A, B and C are the principal moments of inertia of the body; p, q and r are the projections of the angular velocity $\underline{V}$ of the body on the principal axes of inertia; $\underline{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$ is the unit vector in the direction of the Z-axis; M is the mass of the body; g is the gravitational acceleration; x_0, y_0 and z_0 are the coordinates of the center of mass in the moving coordinate system $Oxyz$ The overdot here refers to differentiation with

respect to the time t and $N=(3\lambda/R^3)$ where R is the distance from the fixed point O to the attracting center O_1 and λ is the coefficient of such center.
Equations (1) and (2) admit the following three first integrals

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1,$$
$$Ap\gamma_1+Bq\gamma_2+Cr\gamma_3=C_0,$$
$$(Ap^2+Bq^2+Cr^2)+2Mg(x_0\gamma_1+y_0\gamma_2+z_0\gamma_3)+N(A\gamma_1^2+B\gamma_2^2++C\gamma_3^2)=2C_1,$$
(3)

where C_0 and C_1 are constants.

3. Euler's case

As in Euler's case, we obtain the following first fourth integral according to the presence of Newtonian field

$$A^2p^2+B^2q^2+C^2r^2-N(BC\gamma_1^2+CA\gamma_2^2+AB\gamma_3^2)=C_0^2.$$

(4)

Making use of the first two integrals in (3) and the fourth integral (4), one obtains

$$\gamma_1=\frac{Ap}{C_0},\quad \gamma_2=\frac{Bq}{C_0},\quad \gamma_3=\frac{Cr}{C_0}.$$

(5)

Substituting from systems (1) and (2) into the third equation of system (3), one gets

$$C_2[(N^*A^2+1)Ap^2+(N^*B^2+1)Bq^2+(N^*C^2+1)Cr^2]+x_0Ap+y_0Bq+z_0Cr=C_3,$$

(6)

where

$$C_2=\frac{C_0}{2Mg},\qquad C_3=\frac{C_0C_1}{Mg},\qquad N^*=\frac{N}{C_o^2}.$$

Equation (6) represents a linear combination of the first integrals (3), the fourth integral (4) and (5). So, we seek for a solution that satisfies the previous equation (6).

4. The modified solution

For our scope, let us consider the following choice together with the assumptions of Euler's case
$A>B>C.$

This choice allows us to rewrite equation (6) in the form

$$p^2=\frac{2C_1-N_2^*Bq^2-N_3^*Cr^2}{N_1^*A},$$

(7)

where

$$N_1^*=N^*A^2+1,\quad N_2^*=N^*B^2+1,\quad N_3^*=N^*C^2+1.$$

Substituting from (7) into (4), we can obtain directly q^2 in the form

$$q^2=C_4-C_5r^2.$$

(8)

Here,

$$C_4=\frac{[C_0^2-(2C_1A/N_1^*)+(2C_1N^*ABC/N_1^*)]}{[1-N^*AC-(N_2^*A/N_1^*B)+(N^*N_2^*AC/N_1^*)]B^2},$$
$$C_5=\frac{[1-N^*AB-(N_3^*A/N_1^*C)+(N^*N_3^*AB/N_1^*)]C^2}{[1-N^*AC-(N_2^*A/N_1^*B)+(N^*N_2^*AC/N_1^*)]B^2}.$$

The substitution from (8) into (7) gives

$$p^2=C_6+C_7r^2;$$
$$C_6=(2C_1-N_2^*BC_4)/N_1^*A,$$
$$C_7=(N_2^*BC_5-N_3^*C)/N_1^*A.$$

(9)

Substituting from equalities (8) and (9) into the third one of the system of equations (1), we get

$$\int\frac{dr}{[(A-B)/C]\sqrt{(C_6+C_7)(C_4-C_5)}\ r^2}=\int dt.$$

Under the present circumstances, the solution of the previous integration can be obtained easily as

$$r=\{k-[(A-B)/C]\sqrt{(C_6+C_7)(C_4-C_5)}\ t\}^{-1};k=const.$$

(10)

An inspection of equations (8), (9) and (10), broadly speaking, provides the solution of the problem when the rigid body rotates under the action of a Newtonian force field. This elucidates that, we can separately determine the components of the angular velocity vector p, q and r as functions of time t from these equations. Consequently, we can obtain directly the scalar value of the angular velocity vector in the form

$$V = |\underline{V}| = \left| \sqrt{(C_4 + C_6) + [\frac{1 + C_7 - C_5}{(k - C_8\, t)^2}]} \right|, \qquad (11)$$

where

$C_8 = [(A-B)/C]\sqrt{(C_6 + C_7)(C_4 - C_5)}$.

5. Discussion of results

In this section, our aim is to provide some numerical results using the computer programs. The following data are used to determine the motion in the considered problem

$A = 7\,kg.m^2$, $B = 6\,kg.m^2$, $C = 4\,kg.m^2$, $M = 100\,kg$,

$g = 9.8\,m/s^2$, $N = (200, 400, 500)\,kg.m/s^2$

Figures (1-4) show the variation of the angular velocity V versus time t in 2-D plane when $N = 200 kg.m/s^2$ and $N = 400 kg.m/s^2$. It is to be noted that, the value of the angular velocity of the body monotonically increases with the increase in time (see figures 1, 3) till it has attained its maximum value whenever $t = k/C_8$, i.e. when the dominator of the second bracket in equation (11) vanishes, at different values of Newtonian force field. The domain of equation (11) is $\Re^+ \cup \{0\} - \{k/C_8\}$ and its range is $\Re^+ \cup \{0\}$, where $\Re^+$ is the positive real numbers, $(C_4 + C_6) > 0$ and $(1 + C_7 - C_5) > 0$.

Above the value $t = k/C_8$, the numerical computations show that the angular velocity gradually decreases as the time goes on, in a similar manner to its increase, (see figures 2, 4). Further, we observe that the growth in the value of Newtonian force field leads to increase in t and V as well.

To make the results more favor, we proceed to illustrate the numerical results in 3-D space. Figures (5-7) and (8-10) represent the behavior of the angular velocity V and time t via $\xi = (V - t)$ when the Newtonian force field equals to 200 and 400, respectively. It should be noticed that figures (5, 8), (6, 9) and (7, 10) describe the behavior of the body below, near and above the maximum value of time t, respectively. The spatial figures for most values of Newtonian force field are presented; see figures (11-16).

It is clear from all previous figures that, the Newtonian force field has acquired a significant influence on the behavior of our model. Such results may be utilized in many industrial applications in various fields; like satellite, spacecraft and manipulators.

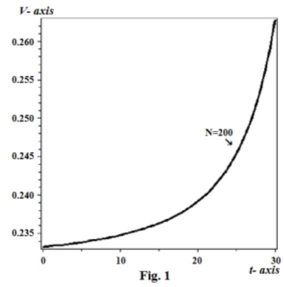

Fig. 1

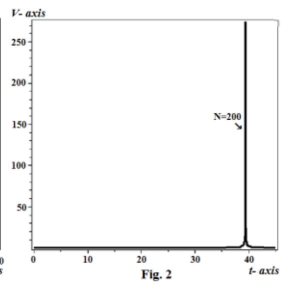

Fig. 2

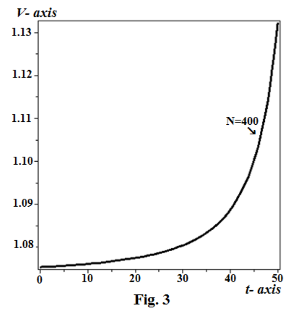

Fig. 3

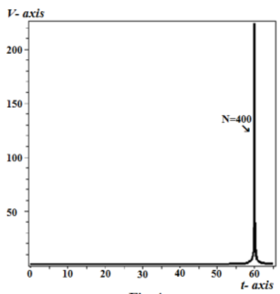

Fig. 4

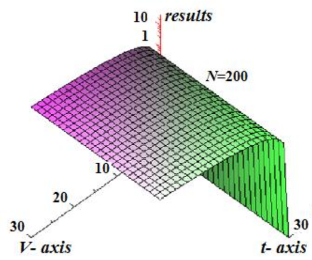

Fig. 5

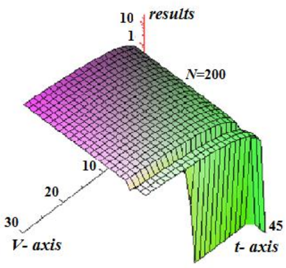

Fig. 6

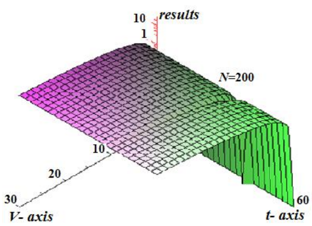

Fig. 7

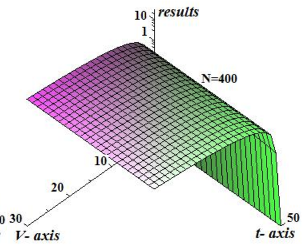

Fig. 8

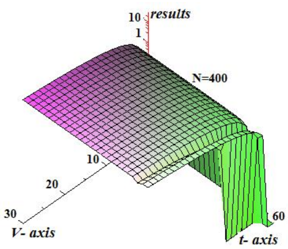

Fig. 9

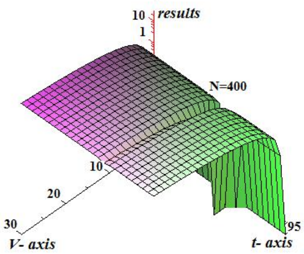

Fig. 10

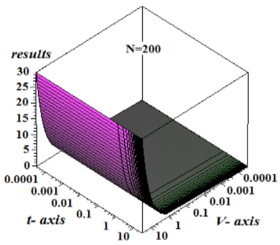

Fig. 11

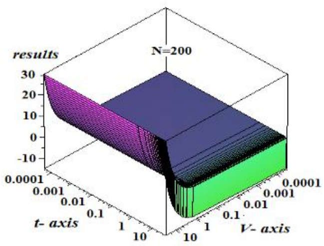

Fig. 12

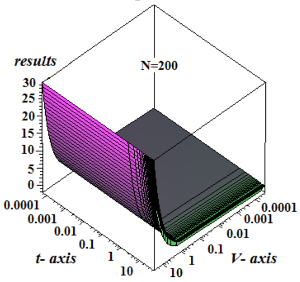

Fig. 13

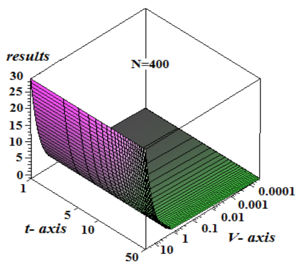

Fig. 14

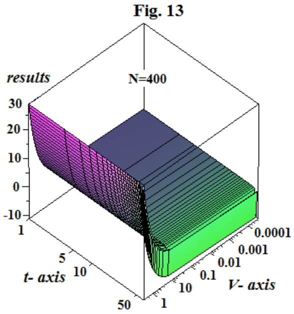

Fig. 15

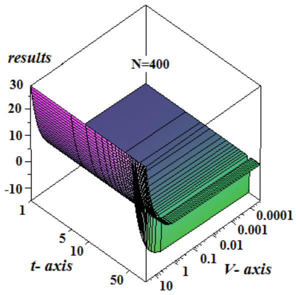

Fig. 16

6. Conclusion

In this work, we have developed a modified solution, represented by (8)-(10), for the Euler's dynamic equations (1) with the aid of Poinsot's equations (2), when the rigid body rotates under the action of a Newtonian force field. The obtained angular velocity components are different from Lagrange's case, Kovaleveskaya's case, Euler's case (when the body rotates without any applied torques) or from any special case. A restriction on the choosing of initial conditions of $p(0), q(0), r(0), \gamma_1(0), \gamma_2(0)$ or $\gamma_3(0)$ according to the meaning of C_0^2 is considered. The obtained solution is considered as a modification for both Euler's case and Ershkov [24] work when the Newtonian filed has no effect, i.e. vanishes. The graphical representations of the obtained angular velocity solution are presented through different figures. The numerical results have shown that the Newtonian force field value has an important effect on the rigid body motion. However, the analytical results of the rotational motion of a rigid body about a fixed point can be exploited in industrial applications, such as satellites, autopilots and aircrafts.

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تقديرات أعلى لحل جسم متماسك ثقيل يتحرك تحت تأثير مجال نيوتوني

في هذا البحث تم دراسة إيجاد حل جديد لمعادلات أويلر الديناميكية للحركة الدورانية للجسم المتماسك حول نقطة ثابتة تحت تأير مجال نيوتيوني، مع الأخذ في الاعتبار أن مركبات متجه السرعة الزاوية الخاصة بهذا الحل تختلف عن معظم الحالات الخاصة المشهورة، وبفرض أن مركز الكتلة للجسم المتماسك ينطبق على نقطة الأصل مع تطبيق شروط إبتدائية معينة. وتم تمثيل الحل الذي تم الوصول إليه هندسياً باستخدام برامج حديثة وذلك لوصف الحركة عند أية لحظة زمنية، ويعتبر هذا الحل تعديلاً لحالة أويلر.

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Mathematics

Associated graphs and chain maps

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Abstract: In this paper, we defined the associated graph constructed to a cellular folding defined on regular CW-complexes. These graphs declare the effect of a cellular folding on the complex. Besides we studied the properties of this graph and we proved that it is connected and vertex transitive if the cellular folding is neat. Finally, by using chain maps and homology groups we obtained the necessary and sufficient conditions for a cellular map to be cellular folding and neat cellular folding respectively.

Key words:

Cellular folding, chain map, regular CW-complexes, vertex transitive, neat folding.

1-Introduction:

The study of foldings of a manifold into anther manifold began with S.A. Robertson's work on isometric folding of Riemannian manifolds [10]. After several attempts of generalizing the notion of isometric foldings, regular foldings were first studies by S.A. Robertson, H.R. Forran and E.El-Kholy [2]. The notion of cellular foldings is invented by E.El-Kholy and H.A.AL-Khurassani [1]. Different types of foldings are introduced by E.EL-Kholy and others [3, 4, 2].

(a) A cell decomposition of a topological space X is a decomposition of X into disjoint open cells such that for each cell e of the decomposition, the boundary $\partial e = \bar{e} - e$ is a union of lower dimensional cells of the decomposition. The set of cells of a cell decomposition of a topological space is called cell complex, [9].

A pair (X, ζ) consisting of a Hausdorff space X and a cell- decomposition ζ of X is called a CW-complex if the following three axioms are satisfied:

1- (Characteristic Maps): For each n-cell $e \in \zeta$ there is a continuous map $\Phi_e : D_n \to X$ restricting to a homeomorphism $\Phi_e\big|_{\text{int}(D_n)} : \text{int}(D_n) \to e$ and taking S^{n-1} into X^{n-1}.

2-(Closure Finiteness): For any cell $e \in \zeta$ the closure $\bar{e}$ intersects only a finite number of other cells in ζ.

3-(Weak Topology): A subset $A \subseteq X$ is closed iff $A \cap \bar{e}$ is closed in X for each $e \in \zeta$, [8].

A <u>CW-complex</u> is said to be regular if all its attaching maps are <u>homemorphisms</u>. If each closed n-cell is homeomorphic to a closed Euclidean n-cell [8]. A topological space that admits the structure of a regular CW-complex is termed a <u>regular CW-space</u>.

(b) Let K and L be cellular complexes and $f : |K| \to |L|$ a continuous map. Then $f : K \to L$ is a cellular map if (i) for each cell $\sigma \in K$, $f(\sigma)$ is a cell in L, (ii) $\dim(f(\sigma)) \le \dim(\sigma)$, [7].

(c) Let K and L be **regular** CW-complexes of the same dimension and K be equipped with finite cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell. A cellular map $f : K \to L$ is a cellular folding iff : (i) for each *i*-cell $\sigma^i \in K, f(\sigma^i)$ is an *i*-cell in L, i.e., *f* maps *i*-cells to *i*-cells, (ii) if $\overline{\sigma}$ contains n vertices, then $\overline{f(\sigma)}$ must contains n distinct vertices.

In the case of directed complexes it is also required that *f* maps directed *i*-cells of K to *i*-cells of L but of the same direction, [5].

A cellular folding $f : K \to L$ is neat if $L^n - L^{n-1}$ consists of a single n-cell, interior L. The set of all cellular foldings of K into L is denoted by *C(K, L)* and the set of all neat foldings of K into L by $\mathcal{N}(K, L)$.

(d) If $f \in C(K, L)$, then $x \in K$ is said to be a singularity of *f* iff *f* is not a local **homeomorphism** at *x*. The set of all singularities of *f* corresponds to the "folds" of the map.

This set associates a cell decomposition C_f of M. If M is a surface, then the edges and vertices of C_f form a graph Γ_f embedded in M, [6].

(e) Let $f : |K| \to |L|$ be a continuous function. If, for each *k*-chain C in K, *f* (C) is a *k*-chain in L and if the diagram

$$\begin{array}{ccc} C_k(K) & \xrightarrow{f} & C_k(L) \\ \partial \downarrow & & \partial \downarrow \\ C_{k-1}(K) & \xrightarrow{f} & C_{k-1}(L) \end{array}$$

commutes, then $f : K \to L$ is a chain function from K to L, [7].

(f) The set S_n of all permutations on n objects forms a group of order *n*!, called the symmetric group of degree n, the law of composition being that for maps of the objects onto themselves. A group of permutations is said to be transitive if, given any pair of letters *a*, *b* (which need not be distinct), there exists at least one permutation in the group which transforms *a* into *b*, [11]. Otherwise the group is called in transitive. And is said to be 1-transitive if for any pair of letters *a*, *b*, there exists a unique element *x* of the group such that $a * x = b$.

<u>2-The associated graph:</u>

Let $f : K \to L$ be a cellular folding. By using the cell subdivision C_f of K we can define the associated graph G_f constructed from the n-cells of K and the cellular folding *f* as follows:

The vertices of G_f are just the n-cells of K and if σ and σ' are distinct n-cells of K such that $f(\sigma) = f(\sigma')$, then there exists an edge E with end points σ and σ'. We then say that E is an edge in G_f with end points σ, σ'.

The graph G_f can be realized as a graph $\tilde{G}_f$ embedded in R^3 as follows. For each n-cells σ, σ' choose any points $v \in \sigma$, $v' \in \sigma'$. If σ and σ' are end points of an edge E, then we can join v to v' by an arc *e* in R^3 that runs from v through σ and σ' to v' crossing *E* transversely at a single point. The correspondence $\sigma \leftrightarrow v$, $E \leftrightarrow e$ is trivially a graph isomorphism from G_f to $\tilde{G}_f$.

It should be noted that the graph G_f has no multiple edges, no loops and generally disconnected.

In this paper by a a complex we mean a regular CW-complex.

<u>Examples(2-1):</u>

(a) Let K be a complex with the cellular subdivisions given in Fig.(1-a). Let $f : K \to K$ be a cellular folding defined by *f* (v_2, v_5, v_8, v_{11}) = (v_4, v_7, v_{10}, v_{13}), *f* (e_1, e_4, e_6, e_9, e_{11}, e_{14}, e_{16}, e_{19}, e_{21}) = (e_3, e_5, e_8, e_{10}, e_{13}, e_{15},

e_{18}, e_{20}, e_{23}) and $f(\sigma_i)=\sigma_{i+1}$, i = 1, 3, 5, 7, 9, where the omitted 0, 1, 2-cells through this paper will be mapped to themselves. The graph G_f in this case has ten vertices and five edges as shown in Fig.(1-b).

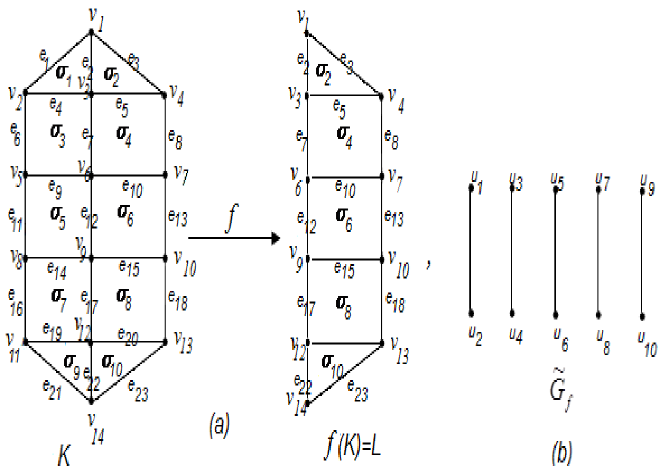

Fig.(1)

b) Consider the complex K shown in Fig.(2), which(consists of one 2-cell, seven 1-cells and seven 0-cells. Let $f:K\to K$ be a cellular folding defined as follow: f (v_5, v_6, v_7) = (v_2, v_3, v_2), f (e_i) = e_2, $i=5,6,7$ and $f(\sigma)=\sigma$. The graph G_f in this case consists of a vertex only with no edges.

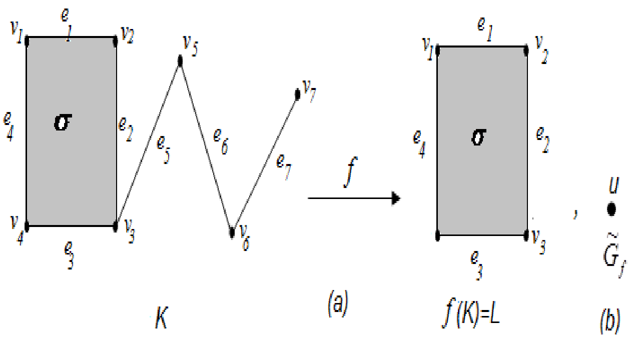

Fig.(2)

(c) Let K be a complex such that $|K|$ is a cylindrical surface with a cellular subdivision consists of eight 0-cells,sixteen 1-cells and eight 2-cells, see Fig.(3). Let $f:K\to K$ be a cellular folding defined by: f (v_5, v_6, v_7, v_8) = (v_1, v_3, v_3, v_3),

$f(e_1,e_2,e_3,e_4,e_5,e_6,e_8,e_{11},e_{12},e_{13},e_{14})=(e_9,e_9,e_9,e_9,$ $e_{15},e_7,e_9,e_{10},e_{16},e_{15},e_{16})$ and $f(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_8)=(\sigma_6,\sigma_6,\sigma_7,\sigma_7,\sigma_6,\sigma_7)$.

This can be done by the composition of the following two cellular foldings: f_1 (v_5, v_8) = (v_1, v_3), $f_1(e_1,e_2,e_6,e_8,e_{11},e_{13},e_{14})=(e_3,e_4,e_7,e_9,e_{10},e_{15},e_{16})$ and $f_1(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=(\sigma_5,\sigma_6,\sigma_7,\sigma_8)$.

$f_2(v_6,v_7)=(v_3,v_3)$, $f_2(e_3,e_4,e_5,e_{12})=(e_9,e_9,e_{15},e_{16})$ and $f_2(\sigma_5,\sigma_8)=(\sigma_6,\sigma_7)$.

The graph G_f in this case has eight vertices and twelve edges see Fig.(3-b).

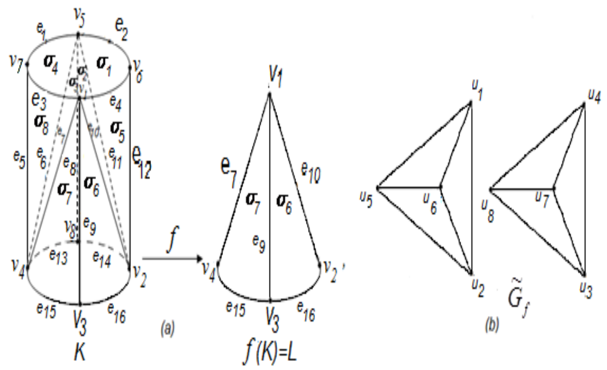

Fig.(3)

(d) Consider a complex K such that $|K|$ is a tours with four 0- cells, eight 1-cells and four 2-cells, see Fig.(4-a). Let $f:K\to K$ be a cellular folding given by: f (v_i) = v_i, i = 1, 2, 3, 4, f (e_3, e_4) = (e_2, e_1) and $f(\sigma_2,\sigma_4)=(\sigma_1,\sigma_3)$. The graph G_f in this case has four vertices and two edges, see Fig.(4-b).

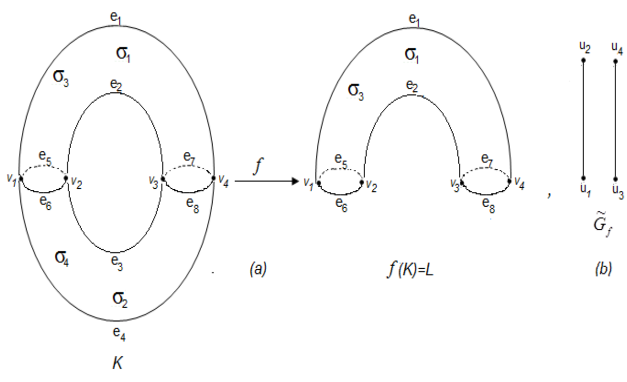

Fig.(4)

3-Properties of the associated graph:

Some of the properties of the associated graph can be characterized by the following theorems:

Theorem (3-1):

Let K and L be complexes of the same dimension n, $f \in C(K,L)$. The associated graph G_f is disconnected unless f is a neat cellular folding.

Proof:

Let σ_1 and σ_2 be distinct n-cells of $K^{(n)}$, and let $\sigma_1 \sim \sigma_2$ means $f(\sigma_1) = f(\sigma_2)$. It is clear that the relation $\sim$ is an equivalence relation. Hence the quotient set $K^{(n)}/\sim$ = $\{[\sigma], \sigma \in K^{(n)}\}$ is a partition on $K^{(n)}$, where $[\sigma]$ is the equivalence class of any n-cell σ. It follows that G_f has more than one component otherwise all the n-cells of K will be mapped to the same n-cell of L which in fact is the case of cellular neat folding. In the last case there will be a unique equivalence class $[\sigma]$ and hence the graph G_f is connected.

It follows from the above theorem that the components of the graph G_f is equal to the number of the equivalence classes generated by the relation ~.

Theorem (3-2):

Let K and L be complexes of the same dimension n, $f \in C(K,L)$ a cellular folding. Then each component of G_f is vertex transitive on itself.

Proof:

From Theorem(3.1) the equivalence relation defined on the n-cells $K^{(n)}$ of K defines a partition $\{[\sigma], \sigma \in K^{(n)}\}$ on $K^{(n)}$, where each equivalence class represents a component of G_f.

Now, consider one of these components G_f^i, with say r vertices, i.e., $|V(G_f^i)| = r$. Each vertex of G_f^i is adjacent to the other vertices in the component, then any permutation of the set $V(G_f^i)$ is an automorphism of G_f^i. Thus the set of all permutations (automorphisms) form a group which is the symmetric group S_r acting on the set $V(G_f^i)$. The orbit of any $\sigma \in V(G_f^i)$ under S_r is the whole set $V(G_f^i)$, i.e., $V(G_f^i)$ has a single orbit and hence the automorphism group S_r is transitive on $V(G_f^i)$.

Results(3-3):

Let $f: K \to L$ be a neat cellular folding:

1) The symmetric group S_r, $r = |K^{(n)}|$ acts 1-transitively on the graph G_f.

2) G_f is vertex transitive.

3) From the above results we conclude that the graph G_f of a neat cellular folding is a complete graph.

Example (3-4):

Consider the complex K shown in Fig.(5-a), which consists of four 2-cells, eight 1-cells and five 0-cells. Let $f: K \to K$ be a cellular folding defined as follows: $f(v_4, v_5) = (v_3, v_2)$, $f(e_4, e_5, e_6, e_7, e_8) = (e_3, e_1, e_2, e_2, e_2)$ and $f(\sigma_i) = \sigma_1$, $i = 1,2,3,4.$ The graph G_f in this case is complete, see Fig(5-b).

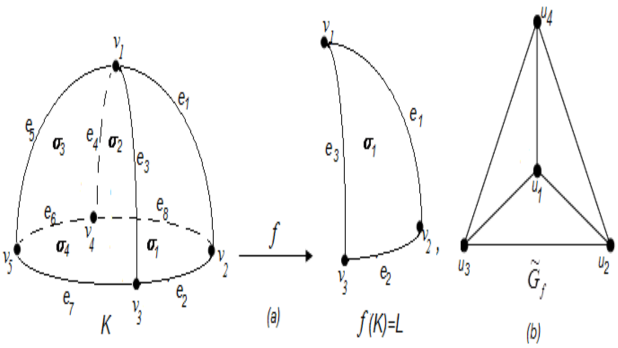

Fig.(5)

(4) Chain maps and cellular folding:

The following theorem gives the necessary and sufficient condition for a cellular map to be a cellular folding.

Theorem(4-1):

Let K and L be complexes of the same dimension n and $f: K \to L$ be a cellular map such that $f(K) = L \neq K.$ Then f

is a cellular folding if and only if the map $f_p : C_p(K) \to C_p(L),$ between chain complexes $(C_p(M), \partial_p)$, $(C_p(N), \partial'_p)$ is a chain map.

Proof:

Let $f : K \to L$ be a cellular folding, then it is a cellular map and for each p-cell $\sigma \in K$ we can define a homomorphism $f_p : C_p(K) \to C_p(L)$ by:

$$f_p = \begin{cases} f(\sigma)\,, & if\ f(\sigma)\ is\ a\ p-cell\ in\ L \\ \varphi\ \ , & if\ \dim(f(\sigma)) < p \end{cases}$$

And since cellular foldings map p-cells to p-cells [5], $f_p(\sigma_\lambda)$ is a p-cell in L for all λ. Thus for a p-chain $C = a_1\sigma_1^p + a_2\sigma_2^p + \ldots + a_k\sigma_k^p \in C_p(K),$ where a_λ'$s \in Z$ and σ_λ's are p-cells in M,

$$a_1 f_p(\sigma_1^p) + a_2 f_p(\sigma_2^p) + \ldots$$
$$f_p(C) = f_p(a_1\sigma_1^p + a_2\sigma_2^p + \ldots + a_k\sigma_k^p) =$$
$$+ a_k f_p(\sigma_k^p) \in C_p(L).$$

Now, since the closures of both σ_λ^p and $f(\sigma_\lambda^p)$ have the same number of distinct vertices, then $f_{p-1} o \partial_p = \partial'_p o f_p$, where $\partial_p : C_p(K) \to C_{p-1}(K)$ and $\partial'_p : C_p(L) \to C_{p-1}(L)$ are the boundary operators, that is to say the following diagram commutes

$$\begin{array}{ccc} C_p(K) & \xrightarrow{f_p} & C_p(L) \\ \downarrow \partial_p & & \downarrow \partial'_p \\ C_{p-1}(K) & \xrightarrow{f_{p-1}} & C_{p-1}(L) \end{array}$$

and hence f_p is a chain map. Conversely, suppose f is not a cellular folding then there exists a j-cell σ in K such that $f(\sigma)$ is an m-cell in L, where $j \neq m$. Since f_p is a homomorphism from the p^{th}-chain of K to the p^{th}-chain of L, then

$$f_j\Big(\sum_{i=1}^{n-1} \lambda_i \sigma_i^{(j)} + \lambda_n \sigma\Big) = \sum_{i=1}^{n-1} \lambda_i f_j(\sigma_i^{(j)}) + \lambda_n f(\sigma),$$

but $f(\sigma)$ is not a j-cell, then f_j cannot be a j-chain map and hence our assumption is false, and we have the result.

Examples (4-2):

(a) Let K be a complex such that $|K|$ is the infinite strip $\{(x,y) : 0 \le x \le \infty,\ 0 \le y \le l\}$ equipped with an infinite number of 2-cells such that the closure of each 2-cell consists of four 0-cells and four 1-cells, P_4. Let L be a complex with six 0-cells, seven 1-cells and two 2-cells, see Fig.(6). The cellular map $f : K \to L$ defined by:

$f(v_i) = v'_i$ where $i = 1,2,\ldots,6$,

$f(v_i) = v'_j$, where $j = 1,2,\ldots,6$ and $(i - j)$ is a multiple of 6,

$f(e_i) = e'_1, i = 1, 11, 21, \ldots, f(e_i) = e'_2, i = 2, 12, 22, \ldots, f(e_i) = e'_3, i = 3, 8, 13, \ldots, f(e_i) = e'_4, i = 4, 9, 14, \ldots, f(e_i) = e'_5, i = 5, 10, 15, \ldots, f(e_i) = e'_6, i = 6, 16, 26, \ldots, f(e_i) = e'_7, i = 7, 17, 27, \ldots$ and $f(\sigma_i) = \begin{cases} \sigma'_1 & , if\ i\ is\ odd, \\ \sigma'_2 & , if\ i\ is\ even \end{cases}$.

is a cellular folding.

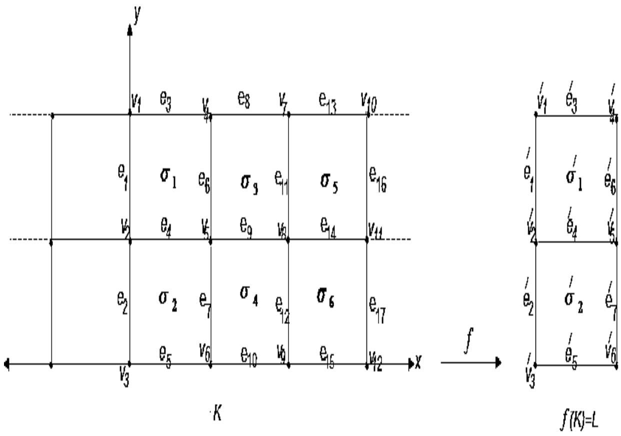

Fig.(6)

(b) Consider a complex K such that $|K| = S^2$, with cellular subdivision consisting of two 0-cells, four 1-cells and four 2-cells. Let $f : K \to K$ be a cellular map defined by: $f(e_2,e_4) = (e_1,e_3)$ and $f(\sigma_i) = \sigma_1, i = 1,...,4.$

This map is a cellular folding with image consisting of two 0-cells, two 1-cells and a single 2-cell, see Fig.(7).

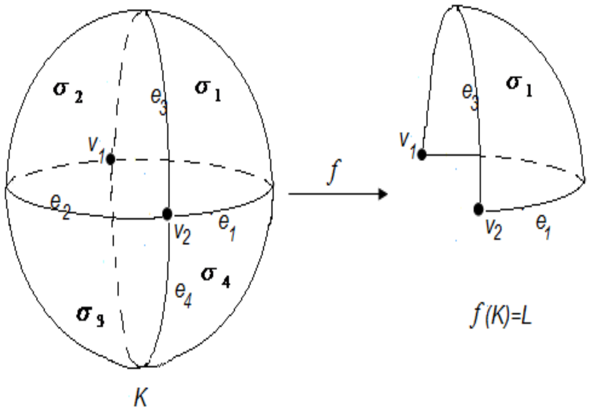

Fig.(7)

(c) Consider a complex K such that $|K|$ is a tours with cellular subdivision consisting of three 0-cells, six 1-cells and three 2-cells. Any cellular map $f : K \to K$ which has two vertices in the image is not a cellular folding since f_1 in this case is not a chain map, see Fig.(8).

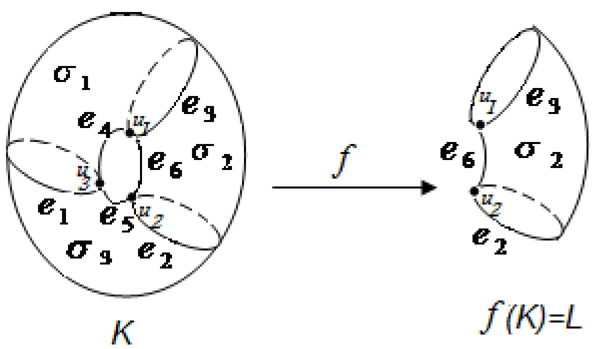

Fig.(8)

(d) Consider a complex K such that $|K| = S^2$, with cellular subdivision consisting of four 0-cells, six 1-cells and four 2-cells, see Fig.(9).

Let $f : K \to K$ be a cellular map defined by $f(v_i) = v_i, i = 1,...,4,$ $f(e_2,e_3) = (e_1,e_4)$ and $f(\sigma_i) = \sigma_2$, $i = 1,...,4.$

This map is not a cellular folding since $\overline{\sigma_1}$ and $\overline{f(\sigma_1)}$ do not contain the same number of vertices.

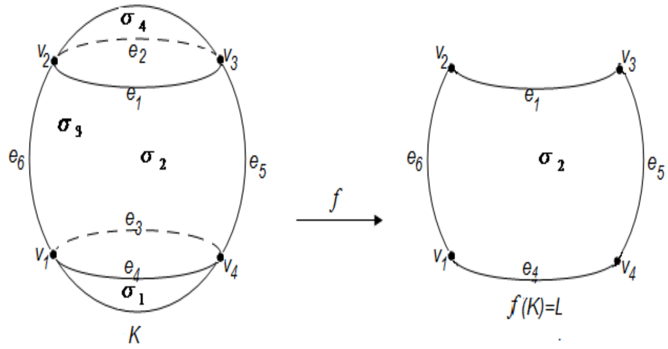

Fig.(9)

Result (4-3):

Let $f : K \to L$, be a cellular folding. Then the induced homomorphism $f_p^* : H_p(K) \to H_p(L)$ will maps the generators of $H_p(K)$ to either the generators of L or to zeros. This follows directly from the fact that the chain map $f_p : C_p(K) \to C_p(L)$ defines a homomorphism that has this property [5].

(5)Homology groups and neat cellular foldings:

The following theorem gives the necessary and sufficient condition for a cellular map to be a neat cellular folding.

Theorem (5-1):

Let K and L be complexes of the same dimension n.

If $f \in C(K,L)$, then f is neat if and only if the map $f_p : C_p(K) \to C_p(L)$ between chain complexes $(C_p(M), \partial_p)$, $(C_p(N), \partial'_p)$ is a chain map and $H_p(K) \cong \ker f_*$, where $f_* : H_p(K) \to H_p(L)$, $p \geq 1$ is the induced homomorphisms.

Proof:

Assuming that f is a neat folding, then it is a cellular folding and hence the map $f_p : H_p(K) \to H_p(L)$ between the chain complexes $(C_p(K), \partial_p), (C_p(L), \partial'_p)$ is a chain map. Now consider the induced homomorphism $f_* : H_p(K) \to H_p(L),$ there is a short exact sequence

$$0 \to \ker f_* \xrightarrow{i^*} H_p(K) \xrightarrow{f_*} \operatorname{Im} f_*$$

where i^* is the induced homomorphism by the inclusion. Since f surjective, we have $\operatorname{Im} f_* \cong H_p(L)$, but $H_p(L) = 0$ for neat cellular foldings, hence the above sequence will take the form

$$0 \to \ker f_* \xrightarrow{i^*} H_p(K) \to 0$$

The exactness of this sequence implies that $H_p(K) \cong \ker f_*$.

Conversely, suppose f is a chain map between chain complexes and $H_p(K) \cong \ker f_*$ but f is not neat, then $L^n - L^{n-1}$ consists of more than one n-cell. Thus $H_0(L) \cong Z^j, H_p(L) = 0,$ for $p = 1,2,...,n$

and $H_p(K) \cong H_p(L) \oplus \ker f_* \cong \ker f_*$

for $p = 0$, and hence the assumption is false and f is neat.

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المخططات المنشأه والدوال السلسلية

فى هذا البحث تم تعريف المخطط المنشأ G_f والمرتبط بالطى الخلوى على التراكيب -CW المنتظمه. هذه المخططات توضح تأثير الطى الخلوى على المركب. بجانب ذلك قدمنا خواص هذا المخطط وأثبتنا إنه مخطط مترابط وله تأثير متعد على الرؤوس (vertex transitive) إذا كان الطى الخلوى صافى. وأخيرا بإستخدام الدوال السلسلية والزمر الهومولوجية حصلنا على الشرط الكافى والضرورى لجعل الدالة الخلويه طى خلوى وطى خلوى صافى على التوالى.

أولا: تم تقديم تعريف المخطط المنشأ مع إعطاء بعض من الأمثلة التى توضح هذا التعريف.

ثانيا: تم توضيح خواص هذا المخطط للطى الخلوى وللطى الصافى على التوالى وأثبتنا التالى:

(١) المخطط المنشأ يكون غير مترابط إلا إذا كان الطى الخلوى هو طى صافى.

(٢) لأى طى خلوى يكون كل مركب من مركبات المخطط المنشأ هو تأثير متعد على رؤوس المركبة.

ثالثا: درسنا حالة أن تكون الدالة الخلويه هى طى خلوى وحصلنا على الشروط المتحققه بواسطة المخططات المنشأة للحصول على الطى المتتابع.

رابعا: درسنا نفس المشكلة ولكن بالنسبه للطى الصافى ولقد حصلنا على الشروط المتحققه بدلالة الزمر الهومولوجية.

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Mathematics

Application to the The truncated distribution of the range for a Wiener process: stock price

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Abstract: In this paper, the exact truncated distribution of the stock price (truncated distribution for the range of a Wiener process) is available among the established results in the field of mathematics (Probability Distributions). Various statistical properties of the distribution are derived including reliability properties, moments, stress-strength parameter, order statistics, Bonferroni curve, Lorenz curve and Gini's index. A real data set is analyzed to clarify the effectiveness of this distribution.

Keywords: Truncated distribution; Wiener process; Reliability properties; Order Statistics.

1. Introduction

Truncation in probability distributions may occur in many studies such as life testing and reliability. Truncation arises because, in many situations, failure of a unit is observed only if it fails before and/or after a certain period. May sometimes happen to be range of the definition of a certain probabilistic distribution is not fully compatible with some of the data, either for theoretical reasons or because the portion of the data cannot be obtained within this range, in this case we resort to the truncated distribution. The truncation method of the distribution is an important methodology in different fields of sciences, in particular communication networks and finance. etc. Truncation occurs in various situations, for example, right truncation occurs in the study of life testing and reliability of items such as an electronic component, light bulbs, etc. Left truncation arises because, in many situations, failure of a unit is observed only if it fails after a certain period. Often, study units may not be followed at the beginning of an experiment until all of them fail, and the experimenter may have to start at a certain time and stop at a certain time when some of the units may still be working. Many researchers were interested in studying the truncation method of the distribution, for example: Zaninetti [12] presents a right and left truncated gamma distribution with application to the stars that introduces an upper and a lower boundary. In addition, the parameters which characterize the truncated gamma distribution are evaluated. A Class of truncated Binomial lifetime distributions is obtained by Alkarni [13]. The type of middle and random truncation have been studied by Mohie El-Din et al. [14] and Teamah et al. [15]. Ali and Nadarajah [3] introduced a truncated version of the Pareto distribution. They derived the explicit expressions for the moments for the truncated version. Nadarajah [4] introduced truncated versions for five of the most commonly known long tailed distributions which possess finite moments of all orders and could therefore be better models. Zaninetti and Ferraro [5] presented a comparison between the Pareto and truncated Pareto distributions. Recently, many papers has been presented the most important applications of the truncated distribution in various fields of science, for example, Pender [7] used the truncated normal distribution to approximate the non stationary single server queue with abandonment. Chattopadhyay et al. [8] provided a more accurate data fitting by using truncated geometric distribution to model the node degree distribution of a network compared to power-law, log-normal, Pareto, drift power-law and power-law with exponential cutoff distributions.

The Wiener process has many applications throughout the mathematical sciences. In physics it is used to study Brownian motion, the diffusion of minute particles suspended in fluid, and other types of diffusion via the Fokker–Planck and Langevin equations. It also forms the basis for the rigorous path integral formulation of quantum mechanics (by the Feynman–Kac formula, a solution to the Schrödinger equation can be represented in terms of the Wiener process) and the study of eternal inflation in physical cosmology. It is also a prominent in the mathematical theory of finance, in particular the Black–Scholes option pricing model. The change of price formula based on the assumption that stock price follow a wiener process. The distribution of stock price through known time interval is the distribution of a Wiener process range. In the time interval $(0,T)$ the range of the Wiener process $\{W(t);t\geq 0\}$ is $\bar{R}(T)=\sup_{(0,T)}W(t)-\inf_{(0,T)}W(t)$ and it gives the difference between the highest price for the stock and it's the lowest price. Feller [1] derived the probability density function of this range by using the method of images. Recently, an expansion for its cumulative distribution function and its quantiles are given by Withers and Nadarajah [2]. In addition, they gave a table of this cumulative distribution function. Here we have the following question: what should be done if we need to find the new distribution of the stock price in the time interval $(0,T)$ and its value is sandwiched between two certain values a, b? To answer the above questions, we should do a truncation on the distribution of a Wiener process range that has been obtained by Feller [1].

In this paper, we will provide the Truncated Distribution of a Wiener Range (TDWR) and study various its statistical properties. The properties studied include reliability properties, moments, stress-strength parameter, order statistics, Bonferroni curve, Lorenz curve and Gini's index. The difference between the TDWR and distribution of a Wiener process range which has been obtained by Feller [1] are showed as in the given figures through the paper.

The paper is organized as follows. In Section 2, we introduce the TDWR. We study some statistical properties for TDWR in Section 3. An application to a real data set is presented in Section 4. Section 5 ends the paper with some concluding remarks and future works.

2. Truncated distribution of a Wiener range (TDWR)

The stock price is assumed to move randomly according to one dimensional Wiener process $\{W(t),t\in\mathrm{R}^{+}\}$, where R^{+} is the set of real numbers and $W(t)$ is a Wiener process on $(0,\infty)$ with range $\bar{R}(T)$ on the time interval $(0,T)$. This range is the difference between $\sup_{(0,T)}W(t)$ and $\inf_{(0,T)}W(t)$. Feller [1] gave the probability density function for the range of $W(t)$ which controls the target's motion as:

$$f_{\bar{R}(T)}(\bar{r})=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\bar{r}^{-1}(2\pi)^{\frac{1}{2}}\left(\frac{\bar{r}T^{-\frac{1}{2}}}{2}\right)^{-1}\times \sum_{k=1}^{\infty}\exp\left[-\frac{(2k-1)^{2}\pi^{2}}{8}.\left(\frac{\bar{r}T^{-\frac{1}{2}}}{2}\right)^{-2}\right] \quad (1)$$

where $0<\bar{r}<\infty$ and $T>0$ and it is represented as in the figure 1.

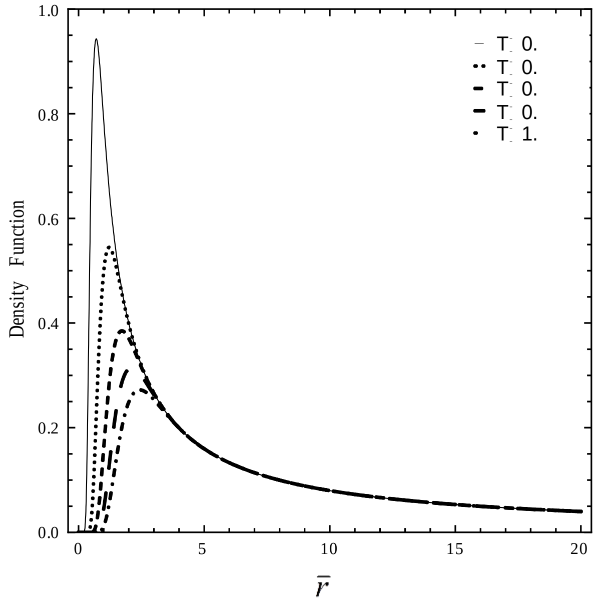

Figure 1: The probability density function of $\bar{R}(T)$.

Withers and Nadarajah [2] give its cumulative distribution function by:

$$F_{\bar{R}(T)}(\bar{r}) = \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + \frac{8T}{\bar{r}^2}\right)\times \exp\left[-\frac{(2k-1)^2\pi^2 T}{2\bar{r}^2}\right] \quad (2)$$

and it is represented as in figure 2.

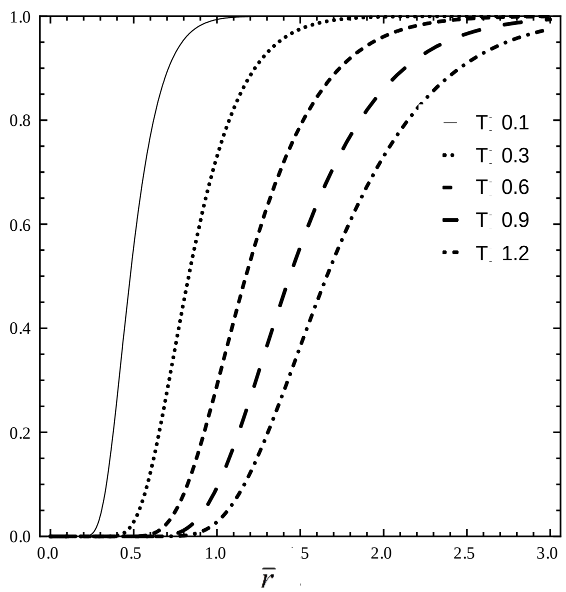

Figure 2: Cumulative distribution function of the range distribution.

Hence, it is easy to show that the survival function $\bar{F}_{\bar{R}(T)}(.) = 1 - F_{\bar{R}(T)}(\bar{r})$ is decreasing by increasing the value of T, see figure 3.

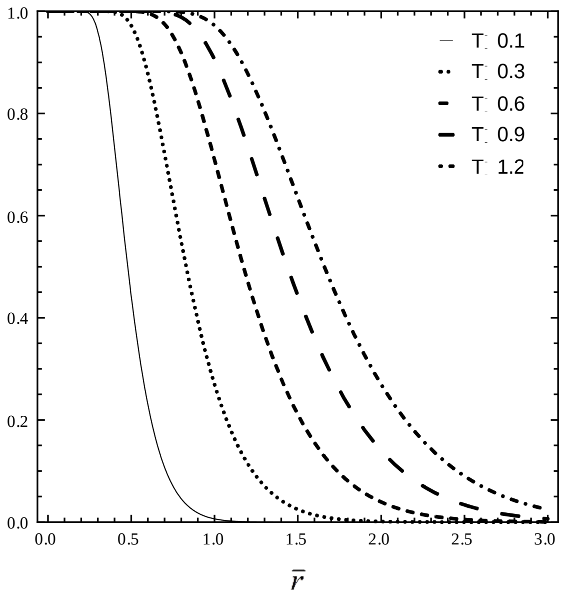

Figure 3: Survival function of the range distribution.

The that defined by (1) lies in its ability to model lifetime data with increasing failure rate.

We are interested in TDWR defined by the following definition.

Definition 1. *Let $\bar{R}(T)$ be a random variable with probability density function (1), define $R(T)$ as a corresponding double truncated (truncation from left and right) of $\bar{R}(T)$ with the probability density function $g_{R(T)}(r)$: Then, the probability density function of double truncated of $\bar{R}(T)$ is given by:*

$$g_{R(T)}(r) = \frac{f_{\bar{R}(T)}(r)}{F_{\bar{R}(T)}(b) - F_{\bar{R}(T)}(a)}$$

$$= \frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8r^{-2}T\right) e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}}\right)}{\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8Tb^{-2}\right) e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} - \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8Ta^{-2}\right) e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}}\right)}$$

(3)

where $a < r < b$ *and* $T > 0$ (see Appendix A) . Figure 4 represents the TDWR density function for different values of a and b with increasing the value of T .

Remark 1. ***Using the ratio test, we can prove that*** $\sum_{k=1}^{\infty} \frac{8e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}}}{(2k-1)^2\pi^2}$ ***is convergent where***

$$\lim_{k\to\infty}\left[\sum_{k=1}^{\infty}\frac{8e^{-\frac{(2(k+1)-1)^2\pi^2 r^{-2}T}{2}}}{(2(k+1)-1)^2\pi^2} \Big/ \sum_{k=1}^{\infty}\frac{8e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}}}{(2k-1)^2\pi^2}\right] = 0.$$

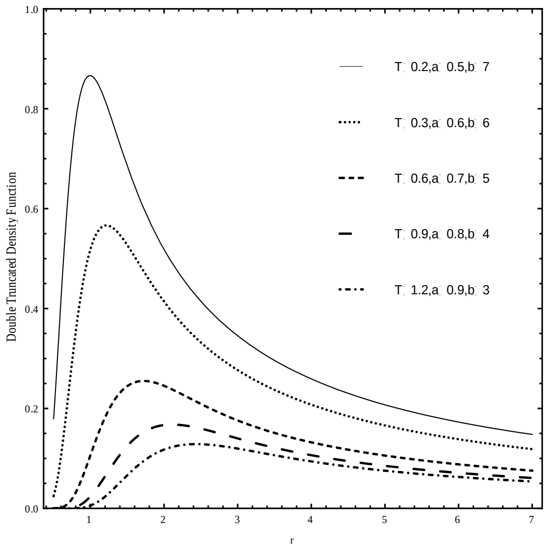

Figure 4: Double truncated probability density

Thus, by the Weierstrass M-Test we see that, $\sum_{k=1}^{\infty} \frac{8e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}}}{(2k-1)^2\pi^2}$ *is uniformly convergent to 0.*

Consequently, we can get (1) from (3) when $a \to 0$ *and* $b \to \infty$

The cumulative distribution function of TDWR is given by:

$$G_{R(T)}(r) = \int_a^r \frac{f_{\bar{R}(T)}(r)}{F_{\bar{R}(T)}(b) - F_{\bar{R}(T)}(a)}\,dr \,,$$

using integration by parts as in the appendix A, we get:

$$G_{R(T)}(r) = \frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty} \frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2 T}{2a^2}} + re^{\frac{-(1-2k)^2\pi^2 T}{2r^2}}\right)}{(\pi - 2k\pi)^2}\right)}{\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} - \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}}\right)},$$

(4)

and it is represented in figure 5.

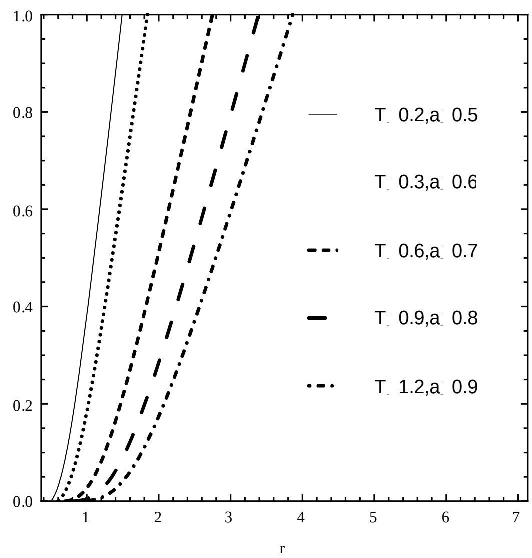

Figure 5: Cumulative distribution function of TDWR.

Consequently, the survival function of TDWR is $\overline{G}_{R(T)}(.) = 1 - G_{R(T)}(r)$ and it is decreasing by increasing the value of T, see figure 6.

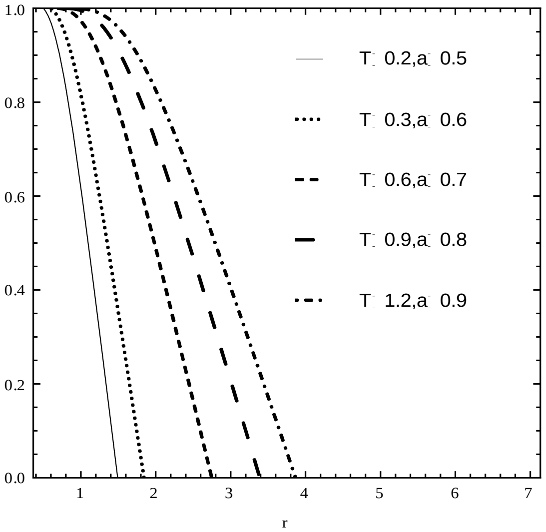

Figure 6: Survival function of the TDWR.

3. Some statistical properties

In this section, we study various statistical properties of the range distribution (truncated and non truncated) including shapes of the probability distribution function and the hazard rate function, reliability properties, raw moments, moments of (reversed) residual life, stress-strength parameter, Bonferroni curve, Lorenz curve and Gini's index.

3.1 Reliability properties

A key concept of "Whenever you want to check more than one investment profits in the stock market, investment whenever exposed to greater risk." You are when you buy or sell shares or bonds or any other financial instruments, you are fair investment risk and the degree of risk this differ from other financial instrument. For example, the financial instruments that you expect them highly profitable (such as active stock) contain a large degree of risk. This means that the share price could rise so much (that is to make a profit for you), but it may happen that the price drops much (and these are the risks that may cause the low volume of your money and your investments). Therefore, the risk rate (hazard rate) is influenced by the swings between fall and rise much of the stock price during the time period $(0,T)$. We get the hazard rate function of the range distribution for Feller [1] and Withers and Nadarajah [2] as follows:

$$z_{\bar{R}(T)}(\bar{r}) = f_{\bar{R}(T)}\{\bar{F}_{\bar{R}(t)}(\bar{r})\}^{-1}$$
$$= \frac{\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \bar{r}^{-1}(2\pi)^{\frac{1}{2}} \left(\frac{\bar{r}T^{-\frac{1}{2}}}{2}\right)^{-1} \sum_{k=1}^{\infty} \exp\left[-\frac{(2k-1)^2\pi^2}{8} \cdot \left(\frac{\bar{r}T^{-\frac{1}{2}}}{2}\right)^{-2}\right]}{1-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+\frac{8T}{\bar{r}^2}\right)\exp\left[-\frac{(2k-1)^2\pi^2 T}{2\bar{r}^2}\right]},$$

(5)

and it is represented as in figure 7. Also, the reversed hazard rate function is:

$$\tilde{z}_{\bar{R}(T)}(\bar{r}) = f_{\bar{R}(T)}\{F_{\bar{R}(t)}(\bar{r})\}^{-1}$$
$$= \frac{\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \bar{r}^{-1}(2\pi)^{\frac{1}{2}} \left(\frac{\bar{r}T^{-\frac{1}{2}}}{2}\right)^{-1} \sum_{k=1}^{\infty} \exp\left[-\frac{(2k-1)^2\pi^2}{8} \cdot \left(\frac{\bar{r}T^{-\frac{1}{2}}}{2}\right)^{-2}\right]}{1-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+\frac{8T}{\bar{r}^2}\right)\exp\left[-\frac{(2k-1)^2\pi^2 T}{2\bar{r}^2}\right]},$$

(6)

See figure 8.

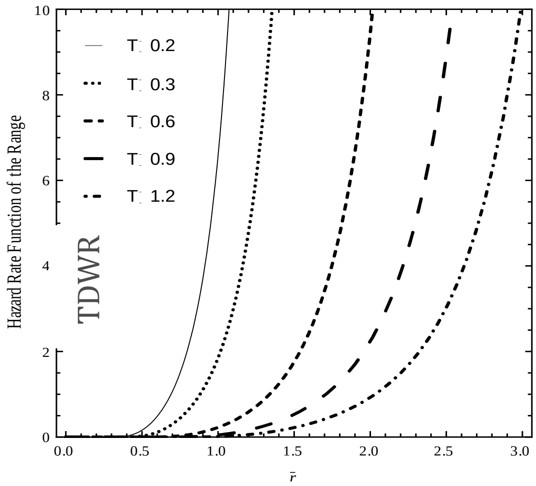

Figure 7: Hazard rate function of the range.

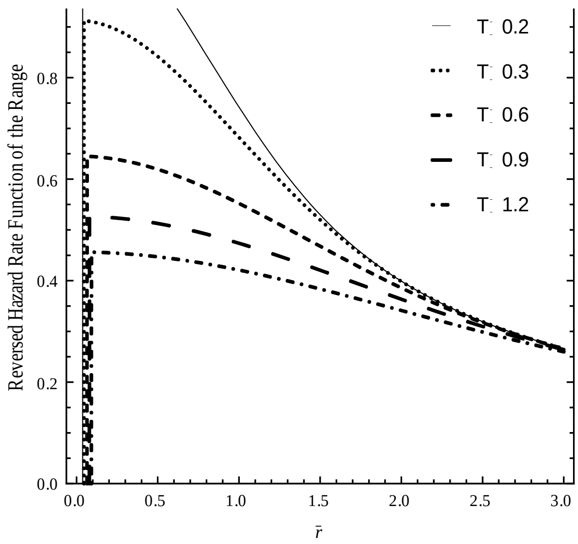

Figure 8: Reversed hazard rate function of the range.

It is clear that, the hazard rate approaches zero as the range $\bar{r}$ increases, and increases rapidly as $\bar{r}$ falls to zero. For the new distribution of TDWR, the hazard rate function is:

$$\Theta_{R(T)}(r) = g_{R(T)}\{\overline{G}_{R(t)}(r)\}^{-1},$$

and it is represented in figure 9.

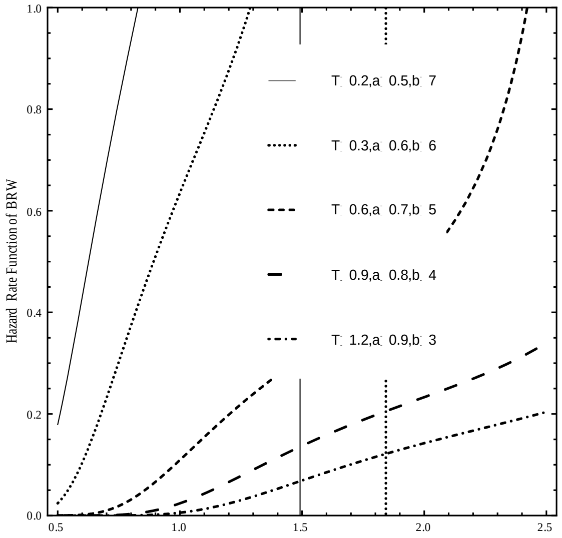

Figure 9: Hazard rate function of TDWR.

In addition, the reversed hazard rate function of TDWR is:

$$\widetilde{\Theta}_{R(T)}(r) = g_{R(T)}\{G_{R(t)}(r)\}^{-1},$$

see figure 10. Also, in this case the hazard rate increases rapidly as r falls to lower bound a.

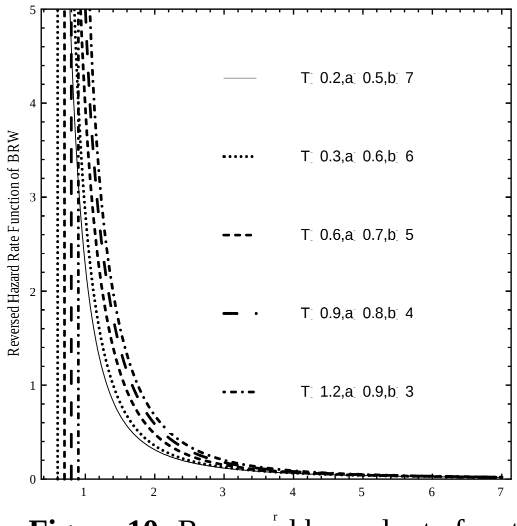

Figure 10: Reversed hazard rate function of TDWR.

3.2 Moments

Many interesting characteristics and features of the range distribution and TDWR can be studied through its generating function and moments. For the range distribution (1) Withers and Nadarajah [2] found its generating, characteristic functions and moments. Here, if R has TDWR distribution and $a < r < b$ and $T > 0$ then the moment generating function (m.g.f.) of R defined by:

$$M(t) = E(e^{tr})$$

$$= \int_a^b \frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty} \left(\frac{8}{(2k-1)^2\pi^2} + 8r^{-2}T \right) e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}} \right) e^{tr}}{\left(\sum_{k=1}^{\infty} \left(\frac{8}{(2k-1)^2\pi^2} + 8Tb^{-2} \right) e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} - \sum_{k=1}^{\infty} \left(\frac{8}{(2k-1)^2\pi^2} + 8Ta^{-2} \right) e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}} \right)} dr$$

$$= \frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right| I(a,b,t,T) \right)}{\left(\sum_{k=1}^{\infty} \left(\frac{8}{(2k-1)^2\pi^2} + 8Tb^{-2} \right) e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} - \sum_{k=1}^{\infty} \left(\frac{8}{(2k-1)^2\pi^2} + 8Ta^{-2} \right) e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}} \right)} \quad (7)$$

where,

$$I(a,b,t,T) = \int_a^b \sum_{k=1}^{\infty} \left(\alpha_k + \frac{\hat{a}}{r^2} \right) e^{\left(tr - \frac{\beta_k}{r^2} \right)} dr, \quad (8)$$

and

$$\alpha_k = \frac{8}{(2k-1)^2\pi^2}, \;\; \hat{a} = 8T, \;\; \beta_k = \frac{(2k-1)^2\pi^2 T}{2}.$$

Since the expansion of the exponential function is valid for $r \in (-\infty, \infty)$ and the series is uniformly convergent, then we have

$$e^{\left(tr - \frac{\beta_k}{r^2} \right)} = \sum_{m=0}^{\infty} \frac{e^{\left(tr - \frac{\beta_k}{r^2} \right)^m}}{m!} = 1 + tr - \frac{\beta_k}{r^2} + \sum_{m=2}^{\infty} \frac{e^{\left(tr - \frac{\beta_k}{r^2} \right)^m}}{m!}$$

$$= 1 + tr - \frac{\beta_k}{r^2} + \sum_{m=2}^{\infty} \sum_{\mu=0}^{m} \frac{t^{m-\mu}}{m!} \binom{m}{\mu} (-\beta_k)^{\mu} r^{m-3\mu}.$$

Assume that,

$I(a,b,t,T) = I_1(a,b,t,T) + I_2(a,b,t,T)$ where,

$$I_1(a,b,t,T)=\sum_{k=1}^{\infty}\int_a^b \alpha_k e^{\left(tr-\frac{\beta_k}{r^2}\right)}dr$$
$$=\sum_{k=1}^{\infty}\int_a^b \alpha_k\left(1+tr-\frac{\beta_k}{r^2}+\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\frac{t^{m-\mu}}{m!}\binom{m}{\mu}(-\beta_k)^{\mu}r^{m-3\mu}\right)dr \tag{9}$$

and

$$I_2(a,b,t,T)=\sum_{k=1}^{\infty}\int_a^b \frac{\hat{a}}{r^2}e^{\left(tr-\frac{\beta_k}{r^2}\right)}dr$$
$$=\hat{a}\sum_{k=1}^{\infty}\int_a^b\left(1+tr-\frac{\beta_k}{r^2}+\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\frac{t^{m-\mu}}{m!}\binom{m}{\mu}(-\beta_k)^{\mu}r^{m-2-3\mu}\right)dr. \tag{10}$$

By solving the following equations:

$$m-3\mu=-1, \tag{11}$$
$$m-2-3\mu=-1, \tag{12}$$

as Diophantine equations, we have the set solution for (11) is given by:

$$S_1=(m_{S_1},\mu_{S_1})=\{(2,1),(5,2),(8,3),(11,4),\ldots\}, \tag{13}$$

and for (12)

$$S_2=(m_{S_2},\mu_{S_2})=\{(4,1),(7,2),(10,3),(13,4),\ldots\}. \tag{14}$$

Hence, $I_1(a,b,t,T)$ can be written as follows:

$$I_1(a,b,t,T)=\sum_{k=1}^{\infty}\alpha_k\left[(b-a)+\frac{t}{2}(b^2-a^2)-\beta_k\left(\frac{1}{a}-\frac{1}{b}\right)\right]$$
$$+\sum_{k=1}^{\infty}\alpha_k\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}t^{m-\mu}}{m!(m+1-3\mu)}\left(b^{m+1-3\mu}-a^{m+1-3\mu}\right)\right]}^{\forall\, m,(\mu\neq\mu_{S_1}\wedge m=m_{S_1})}$$
$$+\sum_{k=1}^{\infty}\alpha_k\left(\sum_{(m,\mu)\in S_1}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}}{m!}t^{m-\mu}\ln\frac{b}{a}\right), \tag{15}$$

Also, $I_2(a,b,t,T)$ can be written as follows:

$$I_2(a,b,t,T)=\hat{a}\sum_{k=1}^{\infty}\left[(a^{-1}-b^{-1})+t\ln\frac{b}{a}-\frac{\beta_k}{3}\left(\frac{1}{a^3}-\frac{1}{b^3}\right)\right]$$
$$+\hat{a}\sum_{k=1}^{\infty}\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}t^{m-\mu}}{m!(m-1-3\mu)}\left(b^{m-1-3\mu}-a^{m-1-3\mu}\right)\right]}^{\forall\, m,(\mu\neq\mu_{S_2}\wedge m=m_{S_2})}$$
$$+\hat{a}\sum_{k=1}^{\infty}\left(\sum_{(m,\mu)\in S_2}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}}{m!}t^{m-\mu}\ln\frac{b}{a}\right). \tag{16}$$

Consequently, from (15) and (16) we obtain the value of (8). Hence,

$$M(t)=E(e^{tr})$$
$$=\int_a^b\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8r^{-2}T\right)e^{-\frac{(2k-1)^2\pi^2r^{-2}T}{2}}\right)e^{tr}}{\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\right)}dr$$
$$=\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\left(I_1(a,b,t,T)+I_2(a,b,t,T)\right)\right)}{\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\right)}. \tag{17}$$

Also, the moments of R about the origin can be obtained as: $E\left(r^q\right)=\int_a^b r^q g_{R(T)}(r)dr.$

$$\tag{18}$$

To solve (18), we let, $x=rT^{-\frac{1}{2}}/2$ and $\omega_k=\left(2k-1\right)^2\pi^2/8$ then (3) become,

$$\Psi_T(x)=\frac{1}{\wp}\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left[\begin{array}{c}(-4x^{-3}\exp(-\omega_k x^{-2}))\\+(2\omega_k x^{-3}\exp(-\omega_k x^{-2}))\\\times(\omega_k^{-1}+2x^{-2})\end{array}\right], \tag{19}$$

where,

$$\wp=\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}.$$

Since the series,

$$\sum_{k=1}^{\infty}\left[\begin{array}{l}(-4x^{-3}\exp(-\omega_k x^{-2}))\\+(2\omega_k x^{-3}\exp(-\omega_k x^{-2}))(\omega_k^{-1}+2x^{-2})\end{array}\right]$$

is uniformly convergence series, then we get,

$$E\left(x^q\right)=\frac{1}{\wp}\sum_{k=1}^{\infty}\Delta_k,$$

where, $\Delta_k=\int_a^b x^q\begin{bmatrix}(-4x^{-3}\exp(-\omega_k x^{-2}))\\+(2\omega_k x^{-3}\exp(-\omega_k x^{-2}))\\\times(\omega_k^{-1}+2x^{-2})\end{bmatrix}dx.$

Thus,

$$\Delta_k=a^{-2+q}\xi\left[\frac{q}{2},\frac{\omega_k}{a^2}\right]-b^{-2+q}\xi\left[\frac{q}{2},\frac{\omega_k}{b^2}\right]+2\omega_k\begin{bmatrix}a^{-4+q}\xi\left[-1+\frac{q}{2},\frac{\omega_k}{a^2}\right]\\+b^{-4+q}\xi\left[-1+\frac{q}{2},\frac{\omega_k}{b^2}\right]\end{bmatrix},$$

where ξ is the exponential integral function. Consequently,

$$E\left(x^q\right)=\frac{\sum_{k=1}^{\infty}\begin{pmatrix}a^{-2+q}\xi\left[\frac{q}{2},\frac{\omega_k}{a^2}\right]-b^{-2+q}\xi\left[\frac{q}{2},\frac{\omega_k}{b^2}\right]\\+2\omega_k\begin{bmatrix}a^{-4+q}\xi\left[-1+\frac{q}{2},\frac{\omega_k}{a^2}\right]\\+b^{-4+q}\xi\left[-1+\frac{q}{2},\frac{\omega_k}{b^2}\right]\end{bmatrix}\end{pmatrix}}{\begin{pmatrix}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}\\-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\end{pmatrix}}$$

By substation again with $x=rT^{-\frac{1}{2}}/2$ and $\omega_k=(2k-1)^2\pi^2/8$ then we get:

$$E\left(r^q\right)=\frac{2^qT^{\frac{q}{2}}\sum_{k=1}^{\infty}\begin{pmatrix}a^{-2+q}\xi\left[\frac{q}{2},\frac{(2k-1)^2\pi^2}{8a^2}\right]\\-b^{-2+q}\xi\left[\frac{q}{2},\frac{(2k-1)^2\pi^2}{8b^2}\right]+\lambda_k\end{pmatrix}}{\begin{pmatrix}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}\\-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\end{pmatrix}},\tag{20}$$

where,

$$\lambda_k=\frac{(2k-1)^2\pi^2}{4}\begin{bmatrix}a^{-4+q}\xi\left[-1+\frac{q}{2},\frac{(2k-1)^2\pi^2}{8a^2}\right]\\+b^{-4+q}\xi\left[-1+\frac{q}{2},\frac{(2k-1)^2\pi^2}{8b^2}\right]\end{bmatrix}.$$

In addition, the characteristic function can get from the equation:

$$\hat{M}(t)=E(e^{itr})$$

$$=\int_a^b\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8r^{-2}T\right)e^{-\frac{(2k-1)^2\pi^2r^{-2}T}{2}}\right)e^{itr}}{\begin{pmatrix}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}\\-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\end{pmatrix}}dr$$

$$=\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\left(\hat{I}_1(a,b,t,T)+\hat{I}_2(a,b,t,T)\right)\right)}{\begin{pmatrix}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}\\-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\end{pmatrix}}\tag{21}$$

where, $i=\sqrt{-1}$,

$$\hat{I}_1(a,b,t,T)=\sum_{k=1}^{\infty}\alpha_k\begin{bmatrix}(b-a)+\frac{it}{2}(b^2-a^2)\\-\beta_k\left(\frac{1}{a}-\frac{1}{b}\right)\end{bmatrix}$$

$$+\sum_{k=1}^{\infty}\alpha_k\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^\mu(it)^{m-\mu}}{m!(m+1-3\mu)}\left(b^{m+1-3\mu}-a^{m+1-3\mu}\right)\right]}^{\forall\ m,(\mu\neq\mu_{S_1}\wedge m=m_{S_1})}$$

$$+\sum_{k=1}^{\infty}\alpha_k\left(\sum_{(m,\mu)\in S_1}\binom{m}{\mu}\frac{(-\beta_k)^\mu}{m!}(it)^{m-\mu}\ln\frac{b}{a}\right),$$

and

$$\hat{I}_2(a,b,t,T)=\hat{a}\sum_{k=1}^{\infty}\begin{bmatrix}(a^{-1}-b^{-1})\\+it\ln\frac{b}{a}-\frac{\beta_k}{3}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\end{bmatrix}$$

$$+\hat{a}\sum_{k=1}^{\infty}\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^\mu(it)^{m-\mu}}{m!(m-1-3\mu)}\left(b^{m-1-3\mu}-a^{m-1-3\mu}\right)\right]}^{\forall\ m,(\mu\neq\mu_{S_2}\wedge m=m_{S_2})}$$

$$+\hat{a}\sum_{k=1}^{\infty}\left(\sum_{(m,\mu)\in S_2}\binom{m}{\mu}\frac{(-\beta_k)^\mu}{m!}(it)^{m-\mu}\ln\frac{b}{a}\right).$$

3.3 Stress-strength parameter

In this section, we find $Y=P(\overline{R}(T_2)<\overline{R}(T_1))$, when $\overline{R}(T_1)$ and $\overline{R}(T_2)$ are two independent random variables distributed as in (1) with T_1, T_2, respectively. In the statistical literature Y is known as the stress-strength parameter which describes the changing of stock price. In addition, Y has a random strength $\overline{R}(T_1)$ that is subject to a random stress $\overline{R}(T_2)$. The changing in stock price at the instant that the stress applied to it exceeds the strength, and the changing will function satisfactorily whenever $\overline{R}(T_2) > \overline{R}(T_1)$; see, for example, Church and Harris [6]. Thus, for the range distribution, Y can be expressed as:

$$Y = \int_0^\infty F_{\overline{R}(T)}(\bar{r};T_2).f_{\overline{R}(T)}(\bar{r};T_1)d\bar{r}.$$

And, for TDWR(T_1) and TDWR(T_2) distribution, we find that $\widetilde{Y} = P(R(T_2) < R(T_1))$ where $\widetilde{Y}$ has a random strength $R(T_1)$ that is subject to a random stress $R(T_2)$.

Consequently, $\widetilde{Y} = \int_a^b G_{R(T)}(r;T_2).g_{R(T)}(r;T_1)dr.$

Now we find $\widetilde{Y}$ by assuming that,

$$Q_1 = \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8T_1 b^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 T b^{-2}}{2}} - \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8T_1 a^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 T_1 a^{-2}}{2}},$$

$$Q_2 = \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8T_2 b^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 T_2 b^{-2}}{2}} - \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8T_2 a^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 T_2 a^{-2}}{2}},$$

$$,C_k = \frac{-8a}{(\pi - 2k\pi)^2},\quad B_k = \frac{8}{(\pi - 2k\pi)^2},$$

$$N_k = \frac{(1-2k)^2\pi^2}{2},\ D_L = \frac{8}{(2L-1)^2\pi^2}\ \text{and}$$

$$F_L = \frac{(2L-1)^2\pi^2}{2}.\ \text{Then,}$$

$$\widetilde{Y} = \frac{\left|\frac{T_1^{-\frac{1}{2}}}{2}\right|\left|\frac{T_2^{-\frac{1}{2}}}{2}\right|}{Q_1 Q_2} \times \sum_{k=1}^{\infty}\sum_{L=1}^{\infty}\int_a^b \left[\begin{matrix}\left(C_k e^{-N_k a^{-2}T_2} + B_k r e^{-N_k r^{-2}T_2}\right)\\ \times\left(D_L e^{-F_L r^{-2}T_1} + 8r^{-2}T_1 e^{-F_l r^{-2}T_1}\right)\end{matrix}\right]dr.$$

Let,

$$J(a,b,T_1,T_2) = \int_a^b \left[\begin{matrix}\left(C_k e^{-N_k a^{-2}T_2} + B_k r e^{-N_k r^{-2}T_2}\right)\\ \times\left(D_L e^{-F_L r^{-2}T_1} + 8r^{-2}T_1 e^{-F_l r^{-2}T_1}\right)\end{matrix}\right]dr$$

$$= \int_a^b \left(\begin{matrix} C_k e^{-N_k a^{-2}T_2} D_L e^{-F_L r^{-2}T_1} \\ + C_k e^{-N_k a^{-2}T_2} 8r^{-2}T_1 e^{-F_l r^{-2}T_1} \\ + B_k r e^{-N_k r^{-2}T_2} D_L e^{-F_L r^{-2}T_1} \\ + B_k r e^{-N_k r^{-2}T_2} 8r^{-2}T_1 e^{-F_l r^{-2}T_1}\end{matrix}\right)dr$$

$$= C_k e^{-N_k a^{-2}T_2} D_L \int_a^b e^{-F_L r^{-2}T_1}dr + C_k e^{-N_k a^{-2}T_2} 8T_1 \int_a^b r^{-2}e^{-F_l r^{-2}T_1}dr + B_k D_L \int_a^b r e^{-N_k r^{-2}T_2} e^{-F_L r^{-2}T_1}dr + B_k 8T_1 \int_a^b r^{-1} e^{-N_k r^{-2}T_2} e^{-F_l r^{-2}T_1}dr\,\cdot$$

$$= C_k e^{-N_k a^{-2}T_2} D_L J_1(a,b,T_1,T_2) + C_k e^{-N_k a^{-2}T_2} 8T_1 J_2(a,b,T_1,T_2) + B_k D_L J_3(a,b,T_1,T_2) + B_k 8T_1 J_4(a,b,T_1,T_2),$$

where,

$$J_1(a,b,T_1,T_2) = \int_a^b e^{-F_L r^{-2}T_1}dr = -a e^{-\frac{F_l T_1}{a^2}} + b e^{-\frac{F_l T_1}{b^2}} + \sqrt{\pi}\sqrt{F_L}\sqrt{T_1}\left(\begin{matrix} - Erf\left(\frac{\sqrt{F_L}\sqrt{T_1}}{a}\right) \\ + Erf\left(\frac{\sqrt{F_L}\sqrt{T_1}}{b}\right)\end{matrix}\right);$$

$$J_2(a,b,T_1,T_2) = \int_a^b r^{-2}e^{-F_l r^{-2}T_1}dr = \frac{\sqrt{\pi}\left(Erf\left(\frac{\sqrt{F_L}\sqrt{T_1}}{a}\right) - Erf\left(\frac{\sqrt{F_L}\sqrt{T_1}}{b}\right)\right)}{2\sqrt{F_L}\sqrt{T_1}};$$

$$J_3(a,b,T_1,T_2) = \int_a^b r e^{-N_k r^{-2}T_2} e^{-F_L r^{-2}T_1}dr = \frac{1}{2}F_L T_1\left(\begin{matrix}\Gamma\left(0, \frac{F_L T_1 + N_K T_2}{a^2}\right) \\ - \Gamma\left(0, \frac{F_L T_1 + N_K T_2}{b^2}\right)\end{matrix}\right)$$

$$+\frac{1}{2}\left(\begin{array}{l} -a^2 e^{-\frac{F_L T_1 + N_K T_2}{a^2}} + b^2 e^{-\frac{F_L T_1 + N_K T_2}{b^2}} \\ + N_K T_2 \left(\begin{array}{l} \Gamma\left(0, \frac{F_L T_1 + N_K T_2}{a^2}\right) \\ -\Gamma\left(0, \frac{F_L T_1 + N_K T_2}{b^2}\right) \end{array}\right) \end{array}\right);$$

and

$$J_4(a,b,T_1,T_2) = \int_a^b r^{-1} e^{-N_k r^{-2} T_2} e^{-F_l r^{-2} T_1} dr$$

$$\frac{1}{2}\left(\begin{array}{l} -\Gamma\left(0, \frac{F_L T_1 + N_K T_2}{a^2}\right) \\ +\Gamma\left(0, \frac{F_L T_1 + N_K T_2}{b^2}\right) \end{array}\right)$$

Consequently,

$$\tilde{Y} = \frac{\left|\frac{T_1^{-\frac{1}{2}}}{2}\right|\left|\frac{T_2^{-\frac{1}{2}}}{2}\right|}{Q_1 Q_2} \sum_{k=1}^{\infty}\sum_{L=1}^{\infty} J(a,b,T_1,T_2)$$

$$= \frac{\left|\frac{T_1^{-\frac{1}{2}}}{2}\right|\left|\frac{T_2^{-\frac{1}{2}}}{2}\right|}{Q_1 Q_2} \sum_{k=1}^{\infty}\sum_{L=1}^{\infty} \left(\begin{array}{l} C_k e^{-N_k a^{-2} T_2} D_L J_1(a,b,T_1,T_2) \\ + C_k e^{-N_k a^{-2} T_2} 8T_1 J_2(a,b,T_1,T_2) \\ + B_k D_L J_3(a,b,T_1,T_2) \\ + B_k 8T_1 J_4(a,b,T_1,T_2) \end{array}\right).$$

(22)

Similarly, we can find

$$Y = \int_0^{\infty} F_{\bar{R}(T)}(\bar{r};T_2).f_{\bar{R}(T)}(\bar{r};T_1) d\bar{r}.$$

3.4 Order statistics

For the European minimum (or maximum) options, Goldman et al. [16] defined and derived the closed form pricing formula. The exact distribution of the maximum and the minimum of the prices-path had been available among the established results in the field of mathematics (Probability Theory). There are several studies in the literature including Bergman [17], Kemna and Vorst [18], Kunitomo and Takahashi [19] and Tumbull and Wakeman [20], they determined the probability distribution of the geometric average of the prices when the underlying asset price follows the log-normal distribution, and the closed form for the option prices were obtained. However, the closed form pricing formula for the arithmetic average options do not seem to be derived yet except for a special case in Bergman [17]. The approximated pricing formula and the algorithms for them are quite well studied. The difficulty seems to be in deriving the exact distribution function of the average price. This make the order statistics are among the most fundamental tools in non-parametric statistics and inference. In this part, we discuss some properties of order statistics for TDWR.

Let $R_{1:n} \le R_{2:n} \le ... \le R_{n:n}$ denote the order statistics of a random sample $R_1, R_2, ..., R_n$ from the TDWR. Then the p.d.f. of the p[th] order statistic $R_{p:n}$ is,

$$g_{R(T)(p:n)} = \frac{n!\left(G_{R(T)}(r)\right)^{p-1}\left(1-G_{R(T)}(r)\right)^{n-p} g_{R(T)}(r)}{(p-1)!(n-p)!},$$

$$= \frac{n!}{(p-1)!(n-p)!}$$

$$\times\left(1-\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2 T}{2a^2}}+re^{\frac{-(1-2k)^2\pi^2 T}{2r^2}}\right)}{(\pi-2k\pi)^2}\right)}{\left(\begin{array}{l}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} \\ -\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}}\end{array}\right)}\right)^{n-p}$$

$$\times\left(\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8r^{-2}T\right)e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}}\right)}{\begin{array}{l}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} \\ -\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}}\end{array}}\right)$$

$$= \frac{\binom{n}{p}\left|\frac{T^{\frac{-1}{2}}}{2}\right|^p}{Q^n}\left(\sum_{k=1}^{\infty}\frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2 T}{2a^2}}+re^{\frac{-(1-2k)^2\pi^2 T}{2r^2}}\right)}{(\pi-2k\pi)^2}\right)^{p-1}$$

$$\times\left(Q-\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2T}{2a^2}}+re^{\frac{-(1-2k)^2\pi^2T}{2r^2}}\right)}{(\pi-2k\pi)^2}\right)^{n-p}$$

$$\times\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8r^{-2}T\right)e^{-\frac{(2k-1)^2\pi^2r^{-2}T}{2}}\right),\qquad(23)$$

where,

$$Q=\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}$$

Also, the distribution function of $R_{p:n}$ is,

$$G_{R(T)(p:n)}(r)=\sum_{i=p}^{n}\binom{n}{i}\left(G_{R(T)}(r)\right)^i\left(1-G_{R(T)}(r)\right)^{n-i}$$

$$=\sum_{i=p}^{n}\binom{n}{i}\left(\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2T}{2a^2}}+re^{\frac{-(1-2k)^2\pi^2T}{2r^2}}\right)}{(\pi-2k\pi)^2}\right)}{\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}\right)}\right)^i$$

$$\times\left(1-\frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2T}{2a^2}}+re^{\frac{-(1-2k)^2\pi^2T}{2r^2}}\right)}{(\pi-2k\pi)^2}\right)}{\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Tb^{-2}}{2}}-\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2}+8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2Ta^{-2}}{2}}}\right)^{n-i}$$

$$=\sum_{i=p}^{n}\frac{\binom{n}{i}\left|\frac{T^{\frac{-1}{2}}}{2}\right|^i}{Q^n}\left(\sum_{k=1}^{\infty}\frac{8\left(-ae^{\frac{-(1-2k)^2\pi^2T}{2a^2}}+re^{\frac{-(1-2k)^2\pi^2T}{2r^2}}\right)}{(\pi-2k\pi)^2}\right)^i$$

$$f:K\to K\qquad(24)$$

In addition, the q[th] moment of the p[th] order statistic $R_{p:n}$ is,

$$E(R_{k:n}^q)=q\sum_{j=n-k+1}^{n}(-1)^{j-n+k-1}\binom{j-1}{n-k}\binom{n}{j}\times\int_a^b r^{q-1}\left(1-F_{R(T)}(r)\right)^j dr$$

Let,

$$\aleph_j=\int_a^b r^{q-1}\left(1-F_{R(T)}(r)\right)^j dr;$$

$$Z_k=\frac{-8a}{(\pi-2k\pi)^2};\text{ and}$$

$$G_K=\frac{(1-2k)^2\pi^2T}{2}.$$

Then,

$$\aleph_j=\int_a^b r^{q-1}\left(1-\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(Z_ke^{-G_ka^{-2}}+8re^{-G_kr^{-2}}\right)\right)^j dr.$$

Also, let $U=\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(Z_ke^{-G_ka^{-2}}+8re^{-G_kr^{-2}}\right).$

From Binomial uniforms theorem we get: $(1-U)^j=\sum_{j=0}^{p}\binom{p}{j}(-1)^jU^j.$

Thus,

$$\aleph_j=\int_a^b r^{q-1}(1-U)^j dr$$

$$= \sum_{j=0}^{p} \binom{p}{j} (-1)^j \int_a^b U^j r^{q-1} dr$$

$$= \sum_{j=0}^{p} \binom{p}{j} (-1)^j$$

$$\times \left(\left| \frac{T^{-\frac{1}{2}}}{2} \right| \right)^j \int_a^b r^{q-1} \left(\sum_{k=1}^{\infty} \begin{pmatrix} Z_k e^{-G_k a^{-2}} \\ +8re^{-G_k r^{-2}} \end{pmatrix} \right)^j dr$$

At $j=0$ we get, $\aleph_0 = \binom{p}{o} \int_a^b r^{q-1} dr = \frac{b^q - a^q}{q}$

. And, at $j=1$ we have,

$$\aleph_1 = \binom{p}{1} (-1) \left| \frac{T^{-\frac{1}{2}}}{2} \right| \int_a^b \sum_{k=1}^{\infty} \begin{pmatrix} Z_k e^{-G_k a^{-2}} \\ +8re^{-G_k r^{-2}} \end{pmatrix} r^{q-1} dr$$

$$= \binom{p}{1} (-1) \left| \frac{T^{-\frac{1}{2}}}{2} \right| \sum_{k=1}^{\infty} \int_a^b \begin{pmatrix} Z_k e^{-G_k a^{-2}} \\ +8re^{-G_k r^{-2}} \end{pmatrix} r^{q-1} dr$$

$$= \binom{p}{1} (-1) \left| \frac{T^{-\frac{1}{2}}}{2} \right|$$

$$\sum_{k=1}^{\infty} \frac{e^{-\frac{G_k}{a^2}} \begin{pmatrix} -4e^{-\frac{G_k}{a^2}} q \begin{pmatrix} a^{1+q} Ei\left(\frac{3+q}{2}, \frac{G_k}{a^2} \right) \\ -b^{1+q} Ei\left(\frac{3+q}{2}, \frac{G_k}{b^2} \right) \end{pmatrix} \\ +\left(-a^q + b^q \right) Z_k \end{pmatrix}}{q},$$

where Ei gives the exponential integral function. Also, at $j=2$ we obtain,

$$\aleph_2 = \binom{p}{2} \left(\left| \frac{T^{-\frac{1}{2}}}{2} \right| \right)^2 \int_a^b \left(\sum_{k=1}^{\infty} \begin{pmatrix} Z_k e^{-G_k a^{-2}} \\ +8re^{-G_k r^{-2}} \end{pmatrix} \right)^2 r^{q-1} dr.$$

Since,

$$\left(\sum_{k=1}^{\infty} \begin{pmatrix} Z_k e^{-G_k a^{-2}} \\ +8re^{-G_k r^{-2}} \end{pmatrix} \right)^2 = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} Z_k e^{-G_k a^{-2}} \\ +8re^{-G_k r^{-2}} \end{pmatrix} \begin{pmatrix} Z_n e^{-G_n a^{-2}} \\ +8re^{-G_n r^{-2}} \end{pmatrix}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} Z_k Z_n e^{-(G_k+G_n) a^{-2}} + 8rZ_k e^{-G_k a^{-2} - G_n r^{-2}} \\ +8rZ_n e^{-G_k r^{-2} - G_n a^{-2}} + 64r^2 e^{-G_k r^{-2} - G_n r^{-2}} \end{pmatrix}.$$

Thus,

$$\aleph_2 = \binom{p}{2} \left(\left| \frac{T^{-\frac{1}{2}}}{2} \right| \right)^2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_a^b \begin{pmatrix} Z_k Z_n e^{-(G_k+G_n) a^{-2}} \\ +8rZ_k e^{-G_k a^{-2} - G_n r^{-2}} \\ +8rZ_n e^{-G_k r^{-2} - G_n a^{-2}} \\ +64r^2 e^{-G_k r^{-2} - G_n r^{-2}} \end{pmatrix} r^{q-1} dr$$

$$= \binom{p}{2} \left(\left| \frac{T^{-\frac{1}{2}}}{2} \right| \right)^2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} Z_k Z_n \int_a^b e^{-(G_k+G_n) a^{-2}} r^{q-1} dr \\ +8Z_K \int_a^b r^q e^{-G_k a^{-2} - G_n r^{-2}} dr \\ +8Z_n \int_a^b r^q e^{-G_k r^{-2} - G_n a^{-2}} \\ dr + 64 \int_a^b r^{q+1} e^{-(G_k+G_n) r^{-2}} dr \end{pmatrix},$$

where,

$$\int_a^b e^{-(G_k+G_n) a^{-2}} r^{q-1} dr = e^{-(G_k+G_n) a^{-2}} \frac{b^q - a^q}{q};$$

$$\int_a^b r^q e^{-G_k a^{-2} - G_n r^{-2}} dr = \frac{1}{2} e^{-\frac{G_k}{a^2}} \begin{pmatrix} -a^{q+1} Ei\left(\frac{3+q}{2}, \frac{G_n}{a^2} \right) \\ +b^{q+1} Ei\left(\frac{3+q}{2}, \frac{G_n}{b^2} \right) \end{pmatrix}$$

and

$$\int_a^b r^{q+1} e^{-(G_k+G_n) r^{-2}} dr = \frac{1}{2} \left(G_k + G_n \right)$$

$$\times \begin{pmatrix} -a^q \Gamma\left(-1 - \frac{q}{2}, \frac{G_k+G_n}{a^2} \right) \left(\frac{a^2}{G_k+G_n} \right)^{-\frac{Q}{2}} \\ +b^q \Gamma\left(-1 - \frac{q}{2}, \frac{G_k+G_n}{b^2} \right) \left(\frac{b^2}{G_k+G_n} \right)^{-\frac{Q}{2}} \end{pmatrix}$$

By the same method we can obtain

$$\aleph_j = \int_a^b r^{q-1} (1-U)^j \, dr.$$

3.5 Bonferroni curve, Lorenz curve and Gini's index

Recently, studies of the stock price has gained a lot of importance. Some important measures in this studies are the Lorenz curve and Gini's index. Lorenzcurve and the associated Gini index are undoubtedly the most popular indices of income in equality. Giorgi and Mondani [9] and Giorgi [10] shown that Bonferroni curve is such a measure, which has the advantage of being represented graphically in the unit square and can also be related to the Lorenz Curve and Gini ratio. Giorgi and Crescenzi [11] presented that these measures have some applications in reliability and life testing as well.

Since R be a non negative random variable with cumulative distribution function (4) which is smooth (i.e., continuous and has derivatives of all orders). However, the first moment of R about zero is finite, exists and non zero as in (20). The Lorenz curve is useful in business modeling: e.g., in consumer finance, to measure the actual percentage of delinquencies attributable to the percentage of people with worst risk scores. Lorenz curve can be obtained by using the equation,

$$L(g_{R(T)}(r)) = \frac{\int_a^r r g_{R(T)}(r)dr}{\int_a^b r g_{R(T)}(r)dr}$$

$$= \frac{\sum_{k=1}^{\infty}\left[-4T\left(\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2a^2}\right]-\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2r^2}\right]\right)+\frac{A}{(\pi-2k\pi)^2}\right]}{\sum_{k=1}^{\infty}\left[-4T\left(\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2a^2}\right]-\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2b^2}\right]\right)+\frac{C}{(\pi-2k\pi)^2}\right]}, \quad (25)$$

where,

$$C = 2\left[-2a^2e^{-\frac{(1-2k)^2\pi^2T}{2a^2}}+2b^2e^{-\frac{(1-2k)^2\pi^2T}{2b^2}} + (1-2k)^2\pi^2T.\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2a^2}\right] - (1-2k)^2\pi^2T.\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2b^2}\right]\right].$$

The Gini index which is defined as a ratio of the areas on the Lorenez curve is given by:

$$G = 1-\frac{2}{Z}\sum_{k=1}^{\infty}\left[-4T(b-a)\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2a^2}\right] + 4T(W+\chi) + \frac{\Lambda+4Y+S-\left(2k(1-2k)^2(1-2k)^2\pi^2T(W+\chi)\right)}{(\pi-2k\pi)^2}\right],$$

where,

$$W = -\sqrt{2}(1-2k)\pi^{\frac{3}{2}}\sqrt{T}Erf\left[\frac{(-1+2k)\pi\sqrt{T}}{\sqrt{2}a}\right] + \sqrt{2}(1-2k)\pi^{\frac{3}{2}}\sqrt{T}Erf\left[\frac{(-1+2k)\pi\sqrt{T}}{\sqrt{2}b}\right];$$

$$\chi = -a\left[-2e^{-\frac{(1-2k)^2\pi^2T}{2a^2}}+\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2a^2}\right]\right] + b\left[-2e^{-\frac{(1-2k)^2\pi^2T}{2b^2}}+\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2b^2}\right]\right];$$

$$Y = -\frac{1}{6}e^{-\frac{(1+4k^2)\pi^2T}{2a^2}}\left[2ae^{\frac{2k\pi^2T}{a^2}}(a^2-(1-2k)^2\pi^2T) - \left[\sqrt{2}e^{-\frac{(1+4k^2)\pi^2T}{2a^2}}(-1+2k)^3\pi^{\frac{7}{2}} \times T^{\frac{3}{2}}Erf\left[\frac{(-1+2k)\pi\sqrt{T}}{\sqrt{2}a}\right]\right]\right]$$

$$+\frac{1}{6}e^{-\frac{(1+4k^2)\pi^2T}{2b^2}}\left[2be^{\frac{2k\pi^2T}{b^2}}(b^2-(1-2k)^2\pi^2T) - \left[\sqrt{2}e^{-\frac{(1+4k^2)\pi^2T}{2b^2}}(-1+2k)^3\pi^{\frac{7}{2}} \times T^{\frac{3}{2}}Erf\left[\frac{(-1+2k)\pi\sqrt{T}}{\sqrt{2}b}\right]\right]\right];$$

$$Z = \sum_{k=1}^{\infty}\left[-4T\left(\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2a^2}\right]-\Gamma\left[0,\frac{(1-2k)^2\pi^2T}{2b^2}\right]\right)+\frac{C}{(\pi-2k\pi)^2}\right];$$

C is given above.

$$S = 2(1-2k)^2(b-a)\pi^2 T.\Gamma\left[0, \frac{(1-2k)^2\pi^2 T}{2a^2}\right];$$

and $\Lambda = -4a^2(b-a)e^{-(1-2k)^2\pi^2 T}$. Also, the Bonferroni curve is given by:

$$B_g(g_{R(T)}(r)) = \frac{L(g_{R(T)}(r))}{G_{R(T)}(r)},$$

where $L(g_{R(T)}(r))$ is given by (22) and from (4) we get $G_{R(T)}(r)$.

4. Application

The oscillation between the fall and rise of the stock price within a time period T can be expressed by a Wiener process. The difference between the highest and the lowest value of the stock price it called the range R of the Wiener process. When the selling price becomes equal to the cost price then $R = 0$ and when the share price up to the upper limit barrier (the upper limit that the stock price has already been reached and reversed to decline) then $R = \infty$. In the upper limit barrier case, the analysts believe that the stock price became expensive and there is no rush to buy it. In this case, sudden drop in the market index may occur while the stock did not reach the point of sale. To avoid a sudden drop in the share price sale we should study the behavior of R by studying some its statistical properties as in Withers and Nadarajah [2]. To ensure that no loss, we should put an upper limit barrier (to avoid sudden drop) and lower limit barrier greater than 0 (a guarantee of a gain even if few). Thus, we use (3) and (4) to get these statistical properties of the bounded range. In [2], Withers and Nadarajah supposed that $x = rT^{-\frac{1}{2}}/2$ where the values of x are given. Here, we let the truncated values of $x_\kappa, \kappa = 1,2,...,5$ for the corresponding time periods are $T_\nu, \nu = 1,2,...,5$, then we get the values of the lower limit barrier a and the upper limit barrier b. Also, we obtain the values of R, the probability density function and cumulative distribution function of R are given in Table 1.

The most important information for the company that builds its decision in order to choose the right time to sell the stock when $a \le R \le b$ is to know the mean value of R.
From (17) we find that the mean value of R is depend on the values of a, b and T as follows:

$$M_1(t) = \frac{\left(\left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty}\left(\frac{4(b^2-a^2)^2}{(2k-1)\pi^2} + 8T\ln\frac{b}{a}\right)\right)}{\left(\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{2}} - \sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{2}}\right)}.$$

Also, the mean values of R for the corresponding time periods $T_\nu, \nu = 1,2,...,5$ are given in Table 1.
By using *mathematica 7*, we found that $\sum_{k=1}^{1000000} \frac{8}{(2k-1)\pi^2} \approx 1$. Thus, in (3), (4) and $M_1(t)$ we get the values of the probability density function, cumulative distribution function and the mean values of R as in Table 1.

Table 1: The probability density function, cumulative distribution function and the mean values of R.

T_v	$a \le r \le b$	r	$g_{R(T)}(r)$	$G_{R(T)}(r)$	$M_1(T)$
		14.14	0.359021	0	
		15.2	0.383331	0.3941	
		15.5	0.3899	0.531761	
50	$14.14 \le r \le 19.799$	15.556	0.389935	0.531762	5.90706×10^7
		16.5	0.404273	0.906967	
		17.3205	0.293429	0	
		17.59	0.298058	0.18806	
		18.2	0.307531	0.264483	
75	$17.3205 \le r \le 24.248$	18.5	0.311711	0.357377	7.23679×10^7
		19.0526	0.318651	0.53159	
		24.495	0.207485	0	
		25.2	0.213385	0.148396	
		26.3	0.221321	0.38762	
150	$24.495 \le r \le 34.2929$	26.944	0.225316	0.531461	1.02351×10^8
		27.2	0.226782	0.589331	
		28.284	0.179672	0	
		29.95	0.189531	0.30787	
		30.12	0.19041	0.340165	
200	$28.284 \le r \le 39.59798$	31.113	0.195119	0.531639	1.1818×10^8
		32.12	0.199212	0.730238	
		31.6228	0.160714	0	
		32.23	0.16382	0.0985405	
		34.785	0.174528	0.531547	
250	$31.6228 \le r \le 44.2719$	35.54	0.177046	0.664283	1.32133×10^8
		36.2	0.179039	0.781801	

5. Concluding remarks

In this paper we introduced a truncated distribution for the range of a Wiener process. This distribution is the best for the stock price in a limited range. We provided a mathematical treatment to find some statistical properties including reliability properties, moments, stress-strength parameter, order statistics, Bonferroni curve, Lorenz curve and Gini's index. A real data set is analyzed to clarify the effectiveness of this distribution. We hope that this distribution may attract a wide applications in lifetime modeling.

In future research one can introduce a new type of middle and random truncation for the range of a Wiener process.

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Appendix A

To get a double truncated density function of TDRW, we put $x = rT^{-\frac{1}{2}}/2$, $\bar{a} = \frac{aT^{-\frac{1}{2}}}{2}$, $\bar{b} = \frac{bT^{-\frac{1}{2}}}{2}$, and $\omega_k = (2k-1)^2\pi^2/8$, then (1) (the probability density function of the range) become:

$$\phi_T(x) = \sum_{k=1}^{\infty}\begin{bmatrix}(-4x^{-3}\exp(-\omega_k x^{-2})) \\ +(2\omega_k x^{-3}\exp(-\omega_k x^{-2})) \\ \times(\omega_k^{-1} + 2x^{-2})\end{bmatrix}.$$

And, its cumulative distribution function is given by:

$$\Phi_T(x) = \sum_{k=1}^{\infty}(\omega_k^{-1} + 2\bar{a}^{-2})\exp(-\omega_k x^{-2})$$

Consequently,

$\Phi_T(\bar{a}) = \sum_{k=1}^{\infty}(\omega_k^{-1} + 2\bar{a}^{-2})\exp(-\omega_k \bar{a}^{-2})$ and

$\Phi_T(b) = \sum_{k=1}^{\infty}(\omega_k^{-1} + 2\bar{b}^{-2})\exp(-\omega_k \bar{b}^{-2})$ .If $\bar{a} \le x \le \bar{b}$ then the double truncated density function can get from the equation:

$$\bar{\phi}_T(x) = \frac{\phi_T(x)}{\Phi_T(\bar{b}) - \Phi_T(\bar{a})} = \frac{\sum_{k=1}^{\infty}\begin{bmatrix}(-4x^{-3}\exp(-\omega_k x^{-2})) \\ +(2\omega_k x^{-3}\exp(-\omega_k x^{-2})) \\ \times(\omega_k^{-1} + 2x^{-2})\end{bmatrix}}{\begin{bmatrix}\sum_{k=1}^{\infty}(\omega_k^{-1} + 2\bar{b}^{-2})\exp(-\omega_k \bar{b}^{-2}) \\ -\sum_{k=1}^{\infty}(\omega_k^{-1} + 2\bar{a}^{-2})\exp(-\omega_k \bar{a}^{-2})\end{bmatrix}}$$

Now by substituent again with $x = \frac{rT^{-\frac{1}{2}}}{2}$, $\bar{a} = \frac{aT^{-\frac{1}{2}}}{2}$, $\bar{b} = \frac{bT^{-\frac{1}{2}}}{2}$, and $\omega_k = (2k-1)^2\pi^2/8$ then the density function of TDWR is given by:

$$\Gamma_{R(T)}(r) = \frac{f_{\bar{R}(T)}(r)}{F_{\bar{R}(T)}(\bar{b}) - F_{\bar{R}(T)}(\bar{a})} = \frac{\left[\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 8r^{-2}T\right)e^{-\frac{(2k-1)^2\pi^2 r^{-2}T}{2}}\right]}{\begin{bmatrix}\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 2Tb^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Tb^{-2}}{8}} \\ -\sum_{k=1}^{\infty}\left(\frac{8}{(2k-1)^2\pi^2} + 2Ta^{-2}\right)e^{-\frac{(2k-1)^2\pi^2 Ta^{-2}}{8}}\end{bmatrix}}.$$

Using integration by parts one can shows that $\int_a^b \Gamma_{R(T)}(r)dr = 1$.

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الملخص العربى

التوزيع المبتور لمدى عمليه وينر: تطبيق على سعر السهم

قسم الرياضيات – كلية العلوم– جامعة طنطا

فى هذه الورقه تم ايجاد توزيع سعر السهم لمنتج فى فتره محدده عن طريق ايجاد التوزيع الاحتمالى المبتور لمدى عملية Wiener العشوائيه. تم ايجاد الخصائص الإحصائية المختلفة للتوزيع بما في ذلك خصائص الموثوقية، العزوم، معامل الاجهاد والقوة ،الإحصاءات الرتيبة، ومنحنى Bonferroni ، منحنى Lorenz ومعامل Gini. تم اعطاء مثال يحتوى على مجموعه مفترضه من البيانات لتوضيح فعالية هذا التوزيع.

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Mathematics

Two Intersection Graphs

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Abstract : We give the number of edges in two intersection graphs.

Keywords : Intersection graph, Stirling numbers.

Introduction

Intersection graph theory is one of the most important topics in graph theory. There is an outstanding concise book titled : "Topics in Intersection Graph Theory" by Terry A. McKee and F. R. McMorris [1], in which the most developed topics of intersection graph theory, emphasizing chordal, interval competition graphs, threshold graphs, p-intersection graphs, intersection multigraphs, pseudographs, and tolerance intersection graphs are discussed. Here we obtaied the number of edges in two intersection graphs, namely : power set intersection graphs and functional intersection graphs. Stirling numbers arise in a varity of analytic and combinatorics problems. We need stirling numbers of the second type in calculating the number of edges of functional intersection graphs. For these numbers the reader is advised to see [2]

1. Power set intersection graph

1.1 Definition. *Let* X := $\{x_1, x_2, \ldots, x_n\}$. Let P(X) be the power set of X, i.e. P(X) = $\{A \mid A \subseteq X\}$.The power set intersection graph is G = (V,E), where V "corresponds to" P (X), and two vertices in V are adjacent if and only if the two corresponding subsets in P(X) have a non-empty intersection.

1.2 Theorem. For a set X := $\{x_1, x_2, \ldots, x_n\}$,the power set intersection graph G = (V,E), has $|V|$, number of vertices = 2^n, and |E|, number of edges = $\frac{1}{2}(4^n - 3^n - 2^n + 1)$

Proof. |V| = 2^n is trivial. Now let $A \subseteq X$, |A| (number of elements of A) = m. The degree of the vertex v_A, which "corresponds to" the set A = $2^n - 2^{n-m} - 1$. It follows that :

$$|E| = \frac{1}{2}\sum_{m=1}^{n}\binom{n}{m}(2^n - 2^{n-m} - 1) =$$

$$\frac{1}{2}.2^n\sum_{m=1}^{n}\binom{n}{m} - \frac{1}{2}.2^n\sum_{m=1}^{n}\binom{n}{m}(\frac{1}{2})^m - \frac{1}{2}\sum_{m=1}^{n}\binom{n}{m}$$

$= \frac{1}{2}.2^n(2^n - 1) \quad - \frac{1}{2}.2^n((1+\frac{1}{2})^n - 1) \quad - \frac{1}{2}(2^n - 1)$

$= 2^{2n-1} - 2^{n-1} - \frac{1}{2}.3^n + \frac{1}{2}$

$= \frac{1}{2}(4^n - 3^n - 2^n + 1)$

1.3 Example. The number of vertices of the power set intersection graph corresponding to the set X := {1,2, 3}, is $2^3 = 8$.

The number of edges = $\frac{1}{2}(4^3 - 3^3 - 2^3 + 1) = 15$

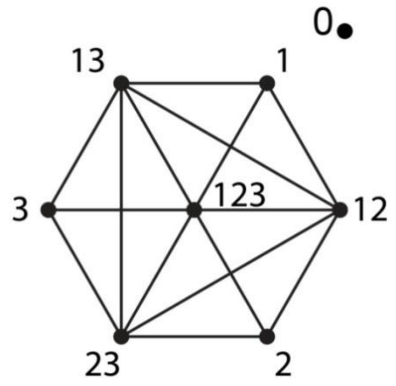

Figure 1 shows such a graph, where the vertex "123" corresponds to the subset {1,2, 3}, the vertex "0" corresponds to the empty set φ. There is an edge joining the vertices "123" and "12", since the subsets {1,2, 3} and {1,2} intersect

2. Functional intersection graph Fig.1

2.1 Definition. *Let* L := $\{f \mid f : X \to Y\}$, be the set of all functions from X into Y. The functional intersection graph G has vertices v_f and v_g "corresponding to" the functions f and g of L. The vertices v_f and v_g are adjacent if and only if range (f) and range (g) have a non-empty intersection.

2.2 Definition. Stirling number of the second kind S_r^m[1] is equal to the number of ways of partitioning a set of m elements into r non-empty subsets,

$$S_r^m = \frac{1}{r!}\sum_{s=1}^{r}(-1)^{r+s}\binom{r}{s}s^m$$

2.3 Remark. The number of all surjective functions from X onto Y, where |X| = m, |Y| = n, $n \le m$ is equal to $n!\ S_n^m$. Consequently, for the set of all functions $f : X \to Y$ having the same range, consisting of r elements, the corresponding is a complete graph consisting of $r!\ S_r^m$ vertices. We note that the number of all functions defined from X into Y

$$= \sum_{r=1}^{n}\binom{n}{r}r!\,S_r^m = n^m,$$

as it is well-known.

2.4 Theorem. The number of edges of the functional intersection graph G corresponding to the set L := $\{f \mid f : X \to Y\}$, whereX := $\{x_1, x_2, \dots, x_m\}$, Y := $\{y_1, y_2, \dots, y_n\}$, is equal to

$$\frac{1}{2}n^m(n^m - 1) - \frac{1}{2}\sum_{r=1}^{n}\binom{n}{r}r!\ S_r^m(n-r)^m$$

$n \le m$

$$\frac{1}{2}n^m(n^m - 1) - \frac{1}{2}\sum_{r=1}^{m}\binom{n}{r}r!\quad S_r^m(n-r)^m$$

m < n

Proof. To explain the situation, we plot every complete subgraph of the same number of vertices in the "same plane", as follows :

$P_1 :\ G_1, G_2, \dots, G_n \Rightarrow \binom{n}{1}$ subgraphs

$P_2 :\ G_{12}, G_{13}, \dots, G_{n-1\,n} \Rightarrow \binom{n}{2}$ subgraphs

$P_3 :\ G_{123}, G_{124}, \dots, G_{n-2\,n-1\,n} \Rightarrow \binom{n}{3}$ subgraphs

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$P_r :\ G_{12\dots r}, \dots \Rightarrow \binom{n}{r}$ subgraphs

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P_{n-2} : $G_{12\ldots n-2}$, … $\Rightarrow \binom{n}{n-2} = \binom{n}{2}$ subgraphs

P_{n-1} : $G_{12\ldots n-1}$, … $\Rightarrow \binom{n}{n-1} = \binom{n}{1}$ subgraphs

P_n : $G_{12\ldots n}$ $\Rightarrow \binom{n}{n} = 1$ subgraph

($G_{12\ldots r}$ is the complete subgraph corresponding to all funcations $f : X \to Y$, X := $\{x_1, x_2, \ldots, x_m\}$, Y := $\{y_1, y_2, \ldots, y_n\}$, their range is $\{1,2,\ldots,r\}$. This complete subgraph consists of $r!\, S_r^m$ vertices, as said before)

Case 1 : $n \le m$

d_1, the degree of any vertex in a subgraph in plane P_1 is given by:

$d_1 = 0+(\binom{n}{2} - \binom{n-1}{2}).\,2!\ S_2^m + (\binom{n}{3} - \binom{n-1}{3}).\,3!\ S_3^m + \ldots$

$+(\binom{n}{n-1} - \binom{n-1}{n-1}).\,(n-1)!\ S_{n-1}^m + \binom{n}{n} n!\ S_n^m$

$= \sum_{\alpha=2}^{n} \binom{n}{\alpha} \alpha!\ S_\alpha^m - \sum_{\beta=2}^{n-1} \binom{n-1}{\beta} \beta!\ S_\beta^m$

$= n^m - n - ((n-1)^m - (n-1))$

$= n^m - (n-1)^m - 1$

d_2, the degree of any vertex in a subgraph in plane P_2 is given by:

$d_2 = 2-1+(\binom{n}{2} - \binom{n-2}{2}).\,2!\ S_2^m + (\binom{n}{3} - \binom{n-2}{3}).\,3!\ S_3^m + \ldots$

$+(\binom{n}{n-2} - \binom{n-2}{n-2}).\,(n-2)!\ S_{n-2}^m + \binom{n}{n-1} (n-1)!\ S_{n-1}^m$

$+\binom{n}{n} n!\ S_n^m$

$= 2 - 1 + \sum_{\alpha=2}^{n} \binom{n}{\alpha} \alpha!\ S_\alpha^m - \sum_{\beta=2}^{n-2} \binom{n-2}{\beta} \beta!\ S_\beta^m$

$= 2 - 1 + n^m - n - ((n-2)^m - (n-2))$

$= n^m - (n-2)^m - 1$

d_r, the degree of any vertex in a subgraph in plane P_r is given by:

$d_r = r-1+(\binom{n}{2} - \binom{n-r}{2}).\,2!\ S_2^m + (\binom{n}{3} - \binom{n-r}{3}).\,3!\ S_3^m + \ldots$

$+(\binom{n}{n-r} - \binom{n-r}{n-r}).\,(n-r)!\ S_{n-r}^m$

$+\binom{n}{n-r+1} (n-r+1)!\ S_{n-r+1}^m + \ldots + \binom{n}{n} n!\ S_n^m$

$d_r = r - 1 + \sum_{\alpha=2}^{n} \binom{n}{\alpha} \alpha!\ S_\alpha^m - \sum_{\beta=2}^{n-r} \binom{n-r}{\beta} \beta!\ S_\beta^m$

$= r - 1 + n^m - n - ((n-r)^m - (n-r))$

$= n^m - (n-r)^m - 1$, $\quad n \le m$

Now the number of all vertices in plane P_r is given by :

$$N_r = \binom{n}{r} r!\ S_r^m,$$

hence $|E|$, the number of edges of the graph G is given by :

$|E| = \frac{1}{2} \sum_{r=1}^{n} N_r . d_r$

$= \frac{1}{2} \sum_{r=1}^{n} \binom{n}{r} r!\ S_r^m ((n^m - 1) - (n-r)^m)$

$= \frac{1}{2} (n^m - 1) \sum_{r=1}^{n} \binom{n}{r} r!\ S_r^m - \frac{1}{2} \sum_{r=1}^{n} (n-r)^m \binom{n}{r} r!\ S_r^m$

$= \frac{1}{2} n^m (n^m - 1) - \frac{1}{2} \sum_{r=1}^{n} \binom{n}{r} r!\ S_r^m (n-r)^m$, $\quad n \le m$ □

Case 2 : m < n .

Here some modifications have to be done. The number of all functions defined from X into Y is given as in case 1 by

$$\sum_{r=1}^{m} \binom{n}{r} r!\ S_r^m = n^m$$

The complete subgraphs in planes P_{m-2} , P_{m-1} , P_m are indicated as follows :

P_{m-2} : $G_{12\ldots m-2}$, … $\Rightarrow \binom{n}{m-2}$ subgraphs

P_{m-1} : $G_{12\ldots m-1}$, … $\Rightarrow \binom{n}{m-1}$ subgraphs

P_m : $G_{12\ldots m}$, … $\Rightarrow \binom{n}{m}$ subgraphs

Now d_1, the degree of any vertex in a subgraph in plane P_1 is given by:

$$d_1 = 0 + \left(\binom{n}{2} - \binom{n-1}{2}\right) 2!\ S_2^m + \left(\binom{n}{3} - \binom{n-1}{3}\right) 3!\ S_3^m + \ldots$$
$$+ \left(\binom{n}{m-1} - \binom{n-1}{m-1}\right)(m-1)!\ S_{m-1}^m$$
$$+ \left(\binom{n}{m} - \binom{n-1}{m}\right) m!\ S_m^m$$

$$d_1 = \sum_{\alpha=1}^{m} \binom{n}{\alpha} \alpha!\ S_\alpha^m - n - \left(\sum_{\beta=1}^{m} \binom{n-1}{\beta} \beta!\ S_\beta^m - (n-1)\right)$$
$$= n^m - n - ((n-1)^m - (n-1))$$
$$= n^m - (n-1)^m - 1$$

To find d_r, the degree of any vertex in a subgraph in plane P_r we have two subcases :

Subcase (i) : n – r < m, here

$$d_r = r - 1 + \left(\binom{n}{2} - \binom{n-r}{2}\right).\ 2!\ S_2^m + \left(\binom{n}{3} - \binom{n-r}{3}\right).\ 3!\ S_3^m + \ldots$$
$$+ \left(\binom{n}{n-r} - \binom{n-r}{n-r}\right).\ (n-r)!\ S_{n-r}^m$$
$$+ \binom{n}{n-r+1} (n-r+1)!\ S_{n-r+1}^m + \ldots$$
$$+ \binom{n}{m} m!\ S_m^m$$
$$= r - 1 + \sum_{\alpha=1}^{m} \binom{n}{\alpha} \alpha!\ S_\alpha^m - n - \left(\sum_{\beta=1}^{n-r} \binom{n-r}{\beta} \beta!\ S_\beta^m - (n-r)\right)$$
$$= r - 1 + n^m - n - ((n-r)^m - (n-r))$$
$$= n^m - (n-r)^m - 1 \quad \text{(the same as in case 1)}$$

Subcase (ii) : n – r ≥ m

$$d_r = r - 1 + \left(\binom{n}{2} - \binom{n-r}{2}\right).\ 2!\ S_2^m + \left(\binom{n}{3} - \binom{n-r}{3}\right).\ 3!\ S_3^m + \ldots$$
$$+ \left(\binom{n}{m} - \binom{n-r}{m}\right) m!\ S_m^m$$
$$= r - 1 + \sum_{\alpha=1}^{m} \binom{n}{\alpha} \alpha!\ S_\alpha^m - n - \left(\sum_{\beta=1}^{m} \binom{n-r}{\beta} \beta!\ S_\beta^m - (n-r)\right)$$
$$= n^m - (n-r)^m - 1 \quad \text{(the same as in case 1)}$$

Now, as before, N_r is the number of vertices in plane P_r which is given by :

$$N_r = \binom{n}{r} r!\ S_r^m ,$$

hence the total number of edges of the graph is given by :

$$|E| = \frac{1}{2} \sum_{r=1}^{m} N_r.\ d_r$$
$$= \frac{1}{2} \sum_{r=1}^{m} \binom{n}{r} r!\ S_r^m \left((n^m - 1) - (n-r)^m\right)$$
$$|E| = \frac{1}{2} (n^m - 1) \sum_{r=1}^{m} \binom{n}{r} r!\ S_r^m - \frac{1}{2} \sum_{r=1}^{m} (n-r)^m \binom{n}{r} r!\ S_r^m$$
$$= \frac{1}{2} n^m (n^m - 1) - \frac{1}{2} \sum_{r=1}^{m} \binom{n}{r} r!\ S_r^m (n-r)^m , \quad m < n \quad \square$$

2.5 Example : X := $\{x_1, x_2, x_3, x_4, x_5\}$, Y := $\{y_1, y_2, y_3\}$

The number of vertices of the corresponding functional intersection graph $= 3^5 = 243$

The number of edges = $\frac{1}{2}$ × 243 × 242 - $\frac{1}{2}\sum_{r=1}^{3}(3-r)^5\binom{3}{r}\ r!\ S_r^5$

= 29403 - $\frac{1}{2}$ ($32 \times 3 \times 1!\ S_1^5 + 3 \times 2!\ S_2^5 +$ 0),

where $S_1^5 = 1$,

$2!\ S_2^5 = \sum_{s=1}^{2}(-1)^{2+s}\binom{2}{s}\ s^5 = -2 + 2^5 = 30$,

hence |E| = 29403 - $\frac{1}{2}$ (96 + 90)

= 29310

2.6 Example : X := $\{x_1, x_2, x_3\}$, Y := $\{y_1, y_2, y_3, y_4, y_5\}$

Number of vertices $= 5^3 = 125$

|E| = number of edges = $\frac{1}{2}$ × 125 × 124 - $\frac{1}{2}\sum_{r=1}^{3}\binom{5}{r}(5-r)^3\ r!\ S_r^3$,

where $S_1^3 = S_3^3 = 1$

$S_2^3 = \frac{1}{2!}\sum_{s=1}^{2}(-1)^{s+2}\binom{2}{s}\ s^3 = 3$

hence |E| = $\frac{1}{2}$ × 125 × 124 - $\frac{1}{2}$(5 ×64+10 ×27×6+10×8×6)

= 7750 – 1210

= 6540

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الملخص العربى

قمنا فى هذا البحث بحساب عدد الأحرف فى شكلين كل منهما عبارة عن رسم تقاطع . ففى الشكل الأول هو رسم تقاطع لقوة فئة عدد عناصرها ن عنصر و الحرف الواصل بين رأسين يعنى أن الفئتين الجزئيتين المناظرتين لهذين الرأسين بينهما تقاطع . و أوجدنا الصيغة التى تعطى عدد هذه الأحرف . و الشكل الثانى هو رسم تقاطع رؤوسه هى المناظرة للدوال المعرفة بين فئتين إحداهما هى المجال و عدد عناصرها م عنصر و الفئة الأخري هى المجال المقابل و عدد عناصرها ن عنصر و الحرف الواصل بين رأسين يعنى أن الدالتين المناظرتين لهذين الرأسين مدى كل منهما بينهما تقاطع و قمنا بحساب عدد الأحرف بالصيغة كما هو موجود بالبحث .

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A proposed method for solving multiobjective linear fractional programming problems with rough coefficients

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Abstract: In this paper, a new method for solving multiobjective linear fractional programming problems with rough coefficient (MORLFP) is proposed. The MORLFP problem is considered by incorporating rough intervals in the coefficients of the objective functions. It is provided that a MORLFP problem is converted to an optimization problem with rough interval valued objective functions which it their bounds are four multiobjective linear fractional functions. The rough efficient solutions are characterized by using a new proposed algorithm. A numerical example is given for the sake of illustration

Key words: Multi objective linear fractional programming problems, Rough interval, Rough interval function.

1. Introduction

Fractional programming concerns with the optimization problems of one or several ratio functions subject to some constraints. Decision makers sometimes, may face up with the decision to optimize actual cost/standard cost, output/employee, etc with respect to some constraints. In management problems, both the ratio functions profit, cost and quality to be optimization are conflicting in nature. Such types of problems are inherently multiobjective fractional programming problems.

Pawlak [11] defined rough set theory as a new mathematical approach to imperfect knowledge. Kryskiewice [8] uses rough set theory to incomplete has found many interesting applications. the rough set approach seems to be of fundamental importance to cognitive sciences, especially in the areas of machine learning, decision analysis, and expert systems Pal [13]. Rough set theory, introduced by Pawlak [12], expresses vagueness, not by means of membership, but employing a boundary region of a set. The theory of rough set deals with the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. Tsumoto [19] used the concept of lower and upper approximation in rough sets theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The concept of rough interval will be introduced by Lu and Huang [9] to represent dual uncertain information of many parameters.The associated solution method will be presented to solve rough interval fuzzy linear programming problems.

Chakraborty and Gupta [3] a different methodology had been proposed for solving multiobjective linear fractional programming (MOLFP) problems always yielding an efficient solution and reduces the complexity in solving the (MOLFP) problems.. Tantawy [18] proposes a new method for solving linear fractional programming problems. Effati and Pakdaman [5] introduce an interval valued linear fractional programming (IVLFP) problem. They convert an IVLFP to an optimization problem with interval valued objective function which its bounds are linear fraction function. Sulaiman and Abulrahim [14] use transformation technique for solving multiobjective linear fractional programming problems to single objective linear fractional programming problem through a new method using mean and median and then solve the problem by modified simplex method. Guzel [6] proposes a new solution to the multiobjective linear fractional programming (MOLFP) problem. Thus MOLFP problem is reduced to linear programming problem. Sulaiman and Abulrahim [17] uses a new transformation technique for solving multiobjective linear fractional programming problems to single objective linear fractional programming problem through a new method using arithmetic average and new arithmetic average technique and then solve the problem by modified simplex method.

This paper deals with a new method for solving MORLFP problem. The MORLFP problem is considered by incorporating rough intervals into coefficient of the objective functions of the problem. The MORLFP problems are converted to four optimization problems. An algorithm is proposed for characterizing the solutions concept of the MORLFP problems. A numerical example is given for the sake of illustration.

2. Preliminaries

2.1 Linear fractional programming problem:

The general linear fractional programming (LFP) problems are defined as follows:

$$\text{Max} \quad \frac{\mathrm{N(x)}}{\mathrm{D(x)}}$$

Subject to:

$x \in X = \{x \in \mathcal{R}^n : \ Ax \le b\ ,\ \ x \ge 0\ \}$,

$c^T,\ d^T \in \mathcal{R}^n\ \alpha,\ \ \beta \in \mathcal{R}\ \ ,\ \ b \in \mathcal{R}^m\ \ ,\ A \in \mathcal{R}^{m\times n}$

Where $N(x) = c^T x + \alpha\ ,\ D(x) = d^T x + \beta$ are real valued and continuous functions on $X\ and\ \ d^T x + \beta\ \neq 0$

Theorem 1. [6] $z^* = \frac{N(x^*)}{D(x^*)} = Max\ \frac{N(x)}{D(x)}$ if and only if

$F(z^*, x^*) = Max\{N(x) - z^* D(x)\ , x \in X\ \} = 0\ .$

2.2 Multi objective linear fractional programming problem

The general multi objective linear fractional programming (MOLFP) problems written as:

$Max\ \ z(x) = \{z_1(x), z_2(x), \ldots\ldots z_k(x)\}$

Subject to:

$x \in X = \{x \in \mathcal{R}^n : \ Ax \le b\ ,\ \ \ x \ge 0\ \}$,

where $z_i(x) = \frac{c_i x + \alpha_i}{d_i x + \beta_i} = \frac{N_i(x)}{D_i(x)}$,

$c_i\ , d_i \in \mathcal{R}^n\ , \alpha_i\ , \beta_i \in \mathcal{R}\ \ D_i(x) > 0,$

$for\ all\ i = 1,2, \ldots, k.$

Definition 1. $x^* \in \mathcal{R}^n$ is an efficient solution for MOLFP problems if there is no $x \in \mathcal{R}^n$ such that $\frac{N_i(x)}{D_i(x)} \geq \frac{N_i(x^*)}{D_i(x^*)}$, $i = 1,2, ...k$ and $\frac{N_i(x)}{D_i(x)} > \frac{N_i(x^*)}{D_i(x^*)}$, for at least one i .

Theorem 2. If $\tilde{x}$ is an optimal solution of

$$Max\{\sum_{i=1}^{k} w_i(N_i(x) - (z_i)^*(D_i(x))),\ x \in X\}$$

where is $(z_i)^* = \frac{N_i(x^*)}{D_i(x^*)} = Max\ \frac{N_i(x)}{D_i(x)}$ for all $i = 1,2, ... k$,

$w_i \in W = \{w_i \in \mathcal{R}^n: w_i \geq 0 , \sum_{i=1}^{k} w_i = 1\}$

then $\tilde{x}$ is an efficient solution of MOLFP problems.

The proof of this theorem is much similar to the proof given by Guzel in [6].

2.3 Rough interval linear fractional programming

Definition 2. Suppose I is the set of all compact intervals in the set of all real numbers $\mathcal{R}$. If $A \in I$ then we write $A = [a^L , a^U]$with $a^L \leq a^U$ and the following holds: [5]

i. $A \geq 0$ iff $x_i \geq 0$ for all $x_i \in A$.

ii. $A \leq 0$ iff $x_i \leq 0$ for all $x_i \in A$.

Definition 3. Let X be denote a compact set of real numbers. A rough interval X^R is defined as:

$X^R = [X^{(LAI)}: X^{(UAI)}]$ where $X^{(LAI)} and\ X^{(UAI)}$ are compact intervals denoted by lower and upper approximation intervals of X^R with $X^{(LAI)} \subseteq X^{(UAI)}$.

Definition 4. For the rough interval X^R the following holds:

i. $X^R \succcurlyeq 0$, iff $X^{(LAI)} \geq 0$ and $X^{(UAI)} \geq 0$

ii. $X^R \preccurlyeq 0$, iff $X^{(LAI)} \leq 0$ and $X^{(UAI)} \leq 0$.

In this paper we denote by I^R is the set of all rough intervals in $\mathcal{R}$. Suppose $A^R , B^R \in I^R$ we can write $A^R = [A^{(LAI)} : A^{(UAI)}]$ and also $B^R = [B^{(LAI)} : B^{(UAI)}]$ where

$A^{(LAI)} = [a^{-L} , a^{+L}]$, $B^{(LAI)}$ $[b^{-L}, b^{+L}]$ a^{-L} , a^{+L}, b^{-L} ,and $b^{+L} \in \mathcal{R}$.

Similarly we can defined $A^{(UAI)}$, $B^{(UAI)}$.

Definition 5. [9] For two rough intervals A^R, B^R when $A^R \succcurlyeq 0\ and\ B^R \succcurlyeq 0$ we can define the following operations on rough intervals as follows:

1) $A^R + B^R = [[A^{(LAI} + B^{(LAI)}] : [A^{(UAI)} + B^{(UAI)}]]$

Such that:

$[A^{(LAI)} + B^{(LAI)}] = [a^{-L} + b^{-L} , a^{+L} + b^{+L}]$ and

$[A^{(UAI)} + B^{(UAI)}] = [a^{-U} + b^{-U} , a^{+U} + b^{+U}]$.

2) $A^R - B^R = [[A^{(LAI} - B^{(LAI)}] : [A^{(UAI)} - B^{(UAI)}]$

Such that:

$[A^{(LAI)} - B^{(LAI)}] = [a^{-L} - b^{+L} , a^{+L} - b^{-L}]$ and

$[A^{(UAI)} - B^{(UAI)}] = [a^{-U} - b^{+U} , a^{+U} - b^{-U}]$.

3) $A^R \times B^R = [\,[A^{(LAI)} \times B^{(LAI)}] : [A^{(UAI)} \times B^{(UAI)}]\,]$

Such that:

$[A^{(LAI)} \times B^{(LAI)}] = [a^{-L} \times b^{-L},\ a^{+L} \times b^{+L}]$ and

$[A^{(UAI)} \times B^{(UAI)}] = [a^{-U} \times b^{-U},\ a^{+U} \times b^{+U}]$.

4) $A^R / B^R = [\,[A^{(LAI)} / B^{(LAI)}] : [A^{(UAI)} / B^{(UAI)}]\,]$

Such that:

$[A^{(LAI)} / B^{(LAI)}] = [a^{-L} / b^{+L},\ a^{+L} / b^{-L}]$ and

$[A^{(UAI)} / B^{(UAI)}] = [a^{-U} / b^{+U},\ a^{+U} / b^{-U}]$.

Definition 6.[5] Let I be the set of all closed and bounded intervals in $\mathcal{R}$.

A function $f: \mathcal{R}^n \to I$ is called an interval valued function with $f(x) = [f^L(x), f^U(x)]$ where for every $x \in \mathcal{R}^n$, $f^L(x)$, $f^U(x)$ are real valued function ,with $f^L(x) \preccurlyeq f^U(x)$.

Definition 7. A function $f: \mathcal{R}^n \to I^R$ is called a rough interval function with

$f^R(x) = [f^{(LAI)}(x) : f^{(UAI)}(x)]$ where for every $x \in \mathcal{R}^n$, $f^{(LAI)}(x)$, $f^{(UAI)}(x)$ are lower and upper approximation interval valued functions, with $f^{(LAI)}(x) \preccurlyeq f^{(UAI)}(x)$

Proposition: [10] Let f be a rough interval function defined on $X \subset \mathcal{R}^n$ and $x_0 \in X$. Then f is continuous at x_0 if and only if $f^{(LAI)}(x)$ and $f^{(UAI)}(x)$ are continuous at x_0

3. Problem Formulation

The multiobjective linear fractional programming problems with rough coefficient (MORLFP) are defined as follows

$$Max\left\{Z_i^R(x) = \frac{N_i^R(x)}{D_i^R(x)} = \frac{c_i^R x + \alpha_i^R}{d_i^R x + \beta_i^R} \quad i = 1,2,\dots k\right\}$$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n : Ax \le b,\ x \ge o\}. \quad (1)$$

where c_i^R, d_i^R, α_i^R and $\beta_i^R \in I^R$, A is an $m \times n$ constraint matrix, $b \in \mathcal{R}^m$, $k \ge 2$.

We can rewrite problem (1) as follows:

$$Max\left\{Z_i^R(x) = \frac{[c_i^{LAI}x+\alpha_i^{LAI} :\ c_i^{UAI}x+\alpha_i^{UAI}]}{[d_i^{LAI}x+\beta_i^{LAI} :\ d_i^{UAI}x+\beta_i^{UAI}]} \quad i = 1,2 \dots, k\right\}$$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n : Ax \le b,\ x \ge o\}. \quad (2)$$

The objective function in (2) is a quotient of two rough interval functions. Using the definition of operations on a rough intervals we have

$$Z_i^R(x) = [\frac{c_i^{LAI}x+\alpha_i^{LAI}}{d_i^{LAI}x+\beta_i^{LAI}} : \frac{c_i^{UAI}x+\alpha_i^{UAI}}{d_i^{UAI}x+\beta_i^{UAI}} \quad i = 1,2 \dots k \quad (3)$$

Now equations (3) can be written into the form:

$Z_i^R(x) = [\ z_i^{LAI}(x) :\ z_i^{UAI}(x)\ ]$ Where $z_i^{LAI}(x)$, $z_i^{UAI}(x)$ lower and upper multiobjective approximation interval valued linear are fractional functions defined as:

$$z_i^{LAI}(x) = \frac{[c_i^{-L}x+\alpha_i^{-L}\ ,\ c_i^{+L}x+\alpha_i^{+L}]}{[d_i^{-L}x+\beta_i^{-L}\ ,\ d_i^{+L}x+\beta_i^{+L}]}$$

and $z_i^{UAI}(x) = \frac{[c_i^{-U}x+\alpha_i^{-U}\ ,\ c_i^{+U}x+\alpha_i^{+U}]}{[d_i^{-U}x+\beta_i^{-U}\ ,\ d_i^{+U}x+\beta_i^{+U}]}$,

for all $i = 1,2 \dots, k$

Using the theorem (2-1) in [5] we can write equation (3) as the following:

$$Z_i^R(x) = \Big[\,[\ z_i^{-L}(x)\ ,\ z_i^{+L}(x)] : [\ z_i^{-U}(x),\ z_i^{+U}(x)]\Big], \qquad (4)$$

where $z_i^{-L}(x)$, $z_i^{+L}(x)$, $z_i^{-U}(x)$ and $z_i^{+U}(x)$, for all $i = 1,2 \dots., k$

are multiobjective linear fractional functions defined as:

$$z_i^{-L}(x) = \frac{c_i^{-L}x+\alpha_i^{-L}}{d_i^{+L}x+\beta_i^{+L}} \quad , \quad z_i^{+L}(x) = \frac{c_i^{+L}x+\alpha_i^{+L}}{d_i^{-L}x+\beta_i^{-L}},$$

$$z_i^{-U}(x) = \frac{c_i^{-U}x+\alpha_i^{-U}}{d_i^{+U}x+\beta_i^{+U}} \quad \text{and} \quad z_i^{+U}(x) = \frac{c_i^{+U}x+\alpha_i^{+U}}{d_i^{-U}x+\beta_i^{-U}}$$

For all $i = 1,2 \dots., k$.

Now the problem (1) can be converted into multiobjective rough interval linear fractional programming (MORLFP) problems as follows:

$$Max\ \{Z_i^R(x) = [\,[\ z_i^{-L}(x),\ z_i^{+L}(x)] : [\ z_i^{-U}(x),\ z_i^{+U}(x)]]\ \},$$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n:\ Ax \le b,\ \ x \ge o\,\}. \quad (5)$$

For all $i = 1,2 \dots., k$

By using the arithmetic operations and partial ordering relations, we decompose the MORLFP problem (5) can be the following four sub problems defines as:

P_1 :

$$Max\ \ z_i^{+U}(x) = \frac{N_i^{+U}(x)}{D_i^{+U}(x)} = \frac{c_i^{+U}x+\alpha_i^{+U}}{d_i^{-U}x+\beta_i^{-U}},\ i = 1,2 \dots, k$$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n:\ Ax \le b,\ \ x \ge o\,\}$$

P_2 :

$$Max\ \ z_i^{-U}(x) = \frac{N_i^{-U}(x)}{D_i^{-U}(x)} = \frac{c_i^{-U}x+\alpha_i^{-U}}{d_i^{+U}x+\beta_i^{+U}},\ i = 1,2 \dots, k$$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n:\ Ax \le b,\ \ x \ge o\,\}$$

$$z_i^{-U}(x) Maximize\ value\ of\ \ z_i^{+U}(x)$$

P_3 :

$$Max\ \ z_i^{+L}(x) = \frac{N_i^{+L}(x)}{D_i^{+L}(x)} = \frac{c_i^{+L}x+\alpha_i^{+L}}{d_i^{-L}x+\beta_i^{-L}}\ i = 1,2 \dots., k$$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n:\ Ax \le b,\ \ x \ge o\,\}$$

$$max\ value\ of\ \ z_i^{-U}(x) \le\ z_i^{+L}(x) \le\ Max\ value\ of\ \ z_i^{+U}(x)$$

P_4 :

$$Max\ \ z_i^{-L}(x) = \frac{N_i^{-L}(x)}{D_i^{-L}(x)} = \frac{c_i^{-L}x+\alpha_i^{-L}}{d_i^{+L}x+\beta_i^{+L}}$$

$, i = 1,2 \dots., k$

Subject to:

$$x \in X = \{x \in \mathcal{R}^n:\ Ax \le b,\ \ x \ge o\,\}$$

$$maximum\ value\ of\ \ z_i^{-U}(x) \le\ z_i^{-L}(x) \le\ Maximize\ value\ of\ \ z_i^{+L}(x)$$

Now using Theorem (2) for socialization problems P_1, P_2, P_3 and P_4 which are MOLFP problems to the equivalent form which are linear programming (LP) problems (P_1', P_2', P_3' and P_4') as follows:

$\mathbf{P_1'}$:

$\{\sum_{i=1}^{k} \omega_i (N_i^{+U}(x) - (Z_i^{+U})^* D_i^{+U}(x)), \ i = 1,2,\dots k \}$

Subject to:

$x \in X = \{x \in \mathcal{R}^n: \ Ax \le b, \ \ x \ge o\}$

$\mathbf{P_2'}$:

$Max \{\sum_{i=1}^{k} \omega_i (N_i^{-U}(x) - (Z_i^{-U})^* D_i^{-U}(x)), i = 1,2,\dots,k \}$

Subject to:

$x \in X = \{x \in \mathcal{R}^n: \ Ax \le b, \ \ x \ge o\}$

$z_i^{-U}(x) \le Maximize\ value\ of\ \ z_i^{+U}(x)$

$\mathbf{P_3'}$:

$$Max \left\{ \sum_{i=1}^{k} \omega_i (N_i^{+L}(x) - (Z_i^{+L})^* D_i^{+L}(x)), i = 1,2,\dots,k \right\}$$

Subject to:

$x \in X = \{x \in \mathcal{R}^n: \ Ax \le b, \ \ x \ge o\}$

maximum value of $z_i^{-U}(x) \le \ z_i^{+L}(x) \ \le$ Maximize value of $z_i^{+U}(x)$

$\mathbf{P_4'}$:

$\text{Max} \{\sum_{i=1}^{k} \omega_i (N_i^{-L}(x) - (Z_i^{-L})^* D_i^{-L}(x)), i = 1,2,\dots,k \}$ Subject to:

$x \in X = \{x \in \mathcal{R}^n: \ Ax \le b, \ \ x \ge o$

$maximum\ value\ of\ \ z_i^{-U}(x) \le \ z_i^{-L}(x) \le Maximize\ value\ of\ \ z_i^{+L}(x)$

Where $\omega \in W = \{\omega_i: \sum_{i=1}^{k} \omega_i = 1, \ \omega_i \ge 0, i = 1,2,\dots,k\}$

Theorem 3.[4] If $x^* \in \mathcal{R}^n$ is an optimal solution for LP problems $P_i', i = 1,2,3,4$ then $x^* \in \mathcal{R}^n$ is an efficient solution of the corresponding MOLFP problems P_i, $i = 1,2,3,4$.

Definition 8. $x^* \in \mathcal{R}^n$ is a rough efficient solution of MORLFP problem (1) if there is no $x \in \mathcal{R}^n$ such that $\frac{N_i^R(x)}{D_i^R(x)} \succcurlyeq \frac{N_i^R(x^*)}{D_i^R(x^*)}$, $i = 1,2,\dots,k$ and $\frac{N_i^R(x)}{D_i^R(x)} \succ \frac{N_i^R(x^*)}{D_i^R(x^*)}$ for at least one i

Theorem 4. If $x^* \in \mathcal{R}^n$ is an efficient solution of the problems P_i, $i = 1,2,3,4$ then $x^* \in \mathcal{R}^n$ is a rough efficient solution of problem (1).

4. Algorithm solution for MORLFP problem :

We construct the algorithm for solving a MORLFP problem as follows:

Step1. Convert the problem to the form of MORLFP problem (5).

Step2. Transfer the problem (5) to four problems on forms P_1, P_2, P_3 and P_4 which are MOLFP problems.

Step3. Find the maximum value of each objective function of P_1, P_2, P_3 and P_4 as:

$$(z_i)^* = \frac{N_i(x^*)}{D_i(x^*)} = \underbrace{\text{Max}}_{x \in X} \ \frac{N_i(x)}{D_i(x)}$$

Step4. Use the weighting method to convert each problems P_1, P_2, P_3 and P_4 to single objective in the form P_1', P_2', P_3' and P_4' respectively.

Step5. Find the optimal solution of each linear programming LP problem P_1', P_2', P_3' and P_4'.

Step6. Using the results of step5, obtain a rough efficient solution to the given MORLFP problem by the Theorem 3 and Theorem 4. with objective value:

$$Z_i^R(x^*) = [[Z_i^{-L}(x^*), Z_i^{+L}(x^*)] : [Z_i^{-U}(x^*), Z_i^{+U}(x^*)]]$$

$for\ all\ i = 1,2 \dots, k$

The algorithm is illustrated with the following example.

5. Numerical example:

Consider the following MORLFP problem:

$$\text{Max } Z_1(x) = \frac{[[1.5,\ 2.5]:[1,3]]x_1 + [[2.5,\ 3.5]:[2,\ 4]]x_2}{[[1,\ 2]:[0.5,\ 3]]x_1 + [[2,\ 3]:[1,\ 5]]x_2 + [[2,\ 5]:[1,\ 7]]},$$

$$Z_2(x) = \frac{[[2,\ 4]:[1,5]]x_1 + [[3,\ 5]:[2,\ 6]]x_2}{[[3,\ 5]:[1,\ 7]]x_1 + [[2,5]:[1,\ 6]]x_2 + [[2,\ 3]:[1,\ 4]]}$$

Subject to:

$x_1 + x_2 \le 5, \quad 3x_1 + x_2 \le 10$

$2x_1 + x_2 \le 7, \quad x_1 \le 3, \quad x_1, x_2 \ge 0$

Now the decomposition problem of the given MORLFP problem as in the following form:

$$\text{Max}\left\{ z_1^{+U}(x) = \frac{3x_1+4x_2}{0.5x_1+x_2+1},\ z_2^{+U}(x) = \frac{5x_1+6x_2}{x_1+x_2+1} \right\}$$

$$\text{Max}\left\{ z_1^{-U}(x) = \frac{x_1+2x_2}{3x_1+5x_2+7},\ z_2^{-U}(x) = \frac{x_1+2x_2}{7x_1+6x_2+4} \right\}$$

$$\text{Max}\left\{ z_1^{+L}(x) = \frac{2.5x_1+3.5x_2}{x_1+2x_2+2},\ z_2^{+L}(x) = \frac{4x_1+5x_2}{3x_1+2x_2+2} \right\}$$

$$\text{Max}\left\{ z_1^{-L}(x) = \frac{1.5x_1+2.5x_2}{2x_1+3x_2+5},\ z_2^{-L}(x) = \frac{2x_1+3x_2}{5x_1+5x_2+3} \right\}$$

Subject to :

$x_1 + x_2 \le 5, \quad 3x_1 + x_2 \le 10$

$2x_1 + x_2 \le 7, \quad x_1 \le 3, \quad x_1, x_2 \ge 0$

Now construct the four problems and solving as follows :

P_1:

$$\text{Ma}\left\{ z_1^{+U}(x) = \frac{3x_1+4x_2}{0.5x_1+x_2+1},\ z_2^{+U}(x) = \frac{5x_1+6x_2}{x_1+x_2+1} \right\}$$

Subject to:

$x_1 + x_2 \le 5, \quad 3x_1 + x_2 \le 10$

$2x_1 + x_2 \le 7, \quad x_1 \le 3, \quad x_1, x_2 \ge 0$

It is observed that $0 \le z_1^{+U} \le 3.71$ and $0 \le z_2^{+U} \le 5$.

This MOLFP problem is equivalent to the following LP problem can be written as:

P_1':

$$\text{Max } \{\omega_1(3x_1 + 4x_2 - 3.71(0.5x_1 + x_2 + 1)) + \omega_2(5x_1 + 6x_2 - 5(x_1 + x_2 + 1))\}$$

Subject to :

$x_1 + x_2 \le 5, \quad 3x_1 + x_2 \le 10$

$2x_1 + x_2 \le 7, \quad x_1 \le 3, \quad x_1, x_2 \ge 0$

For $\omega_1 = \omega_2 = 0.5$

The optimal solution of the LP problem P_1' is obtained as: $x_1^* = 0$, $x_2^* = 5$

The efficient solution of MOLFP problem P_1 are:

$x_1^* = 0$, $x_2^* = 5$ with objective value

. $z_1^{+U} = 3.33$, $z_2^{+U} = 5$.

P_2:

$$\text{Max}\left\{ z_1^{-U}(x) = \frac{x_1+2x_2}{3x_1+5x_2+7} \ ,\ z_2^{-U}(x) = \frac{x_1+2x_2}{7x_1+6x_2+4}\right\}$$

Subject to:

$$\frac{x_1+2x_2}{3x_1+5x_2+7} \leq 3.33\,, \quad \frac{x_1+2x_2}{7x_1+6x_2+4} \leq 5\,,$$

$$x_1 + x_2 \leq 5, \quad 3x_1 + x_2 \leq 10$$

$$2x_1 + x_2 \leq 7, \quad x_1 \leq 3\,, \quad x_1,\ x_2 \geq 0$$

It is observed that $0 \leq z_1^{-U} \leq 0.31$ and $0 \leq z_2^{-U} \leq 0.29$.

This MOLFP problem is equivalent to the following LP problem can be written as:

P_2':

$$\text{Max}\left\{\omega_1\big(x_1 + 2x_2 - 0.31(3x_1 + 5x_2 + 7)\big) + \omega_2\,(x_1 + 2x_2 - 0.29(7x_1 + 6x_2 + 4))\right\}$$

Subject to:

$$\frac{x_1+2x_2}{3x_1+5x_2+7} \leq 3.33\,, \quad \frac{x_1+2x_2}{7x_1+6x_2+4} \leq 5$$

$$x_1 + x_2 \leq 5, \quad 3x_1 + x_2 \leq 10$$

$$2x_1 + x_2 \leq 7, \quad x_1 \leq 3\,, \quad x_1,\ x_2 \geq 0$$

For $\omega_1 = \omega_2 = 0.5$

The optimal solution of the LP problem P_2' is obtained as: $x_1^* = 0$, $x_2^* = 5$

The efficient solution of MOLFP problem P_2 are:

$x_1^* = 0$, $x_2^* = 5$ with the objective value $z_1^{-U} = 0.31$, $z_2^{-U} = 0.29$.

P_3 :

$$\text{Max}\left\{ z_1^{+L}(x) = \frac{2.5x_1+3.5x_2}{x_1+2x_2+2},\ z_2^{+L}(x) = \frac{4x_1+5x_2}{3x_1+2x_2+2}\right\}$$

Subject to :

$$0.31 \leq \frac{2.5x_1 + 3.5x_2}{x_1 + 2x_2 + 2} \leq 3.33\,,$$

$$0.29 \leq \frac{4x_1+5x_2}{3x_1+2x_2+2} \leq 5\,,$$

$$x_1 + x_2 \leq 5\,, \quad 3x_1 + x_2 \leq 10\,,$$

$$2x_1 + x_2 \leq 7, \quad x_1 \leq 3\,, \quad x_1, x_2 \geq 0$$

It is observed that $0 \leq z_1^{+L} \leq 1.57$ and $0 \leq z_2^{+L} \leq 2.08$.

This MOLFP problem is equivalent to the following LP problem can be written as:

P_3' :

$$\text{Max}\left\{\omega_1\big(2.5\,x_1 + 3.5x_2 - 1.57(x_1 + 2x_2 + 2)\big) + \omega_2\,(4x_1 + 5x_2 - 2.08(3\,x_1 + 2x_2 + 2))\right\},$$

Subject to :

$$0.31 \leq \frac{2.5x_1+3.5x_2}{x_1+2x_2+2} \leq 3.33\,,$$

$$0.29 \leq \frac{4x_1+5x_2}{3x_1+2x_2+2} \leq 5 \quad ,$$

$$x_1 + x_2 \leq 5\,, \quad 3x_1 + x_2 \leq 10\,,$$

$$2x_1 + x_2 \leq 7, \quad x_1 \leq 3\,, \quad x_1, x_2 \geq 0\ .$$

For $\omega_1 = \omega_2 = 0.5$

The optimal solution of the LP problem P_3' is obtained as: $x_1^* = 0$, $x_2^* = 5$

The efficient solution of MOLFP problem P_3 are $x_1^* = 0$, $x_2^* = 5$, with objective value $z_1^{+L} = 1.46$, $z_2^{+L} = 2.08$

P_4 :

$\text{Max}\left\{ z_1^{-L}(x) = \frac{1.5x_1+2.5x_2}{2x_1+3x_2+5} \;,\; z_2^{-L}(x) = \frac{2x_1+3x_2}{5x_1+5x_2+3}\right\}$

Subject to :

$0.31 \le \frac{1.5x_1+2.5x_2}{2x_1+3x_2+5} \le 1.46$,

$0.29 \le \frac{2x_1+3x_2}{5x_1+5x_2+3} \le 2.08$,

$x_1 + x_2 \le 5 \quad 3x_1 + x_2 \le 10$,

. $2x_1 + x_2 \le 7, \quad x_1 \le 3 , \quad x_1 , x_2 \ge 0$

It is observed that $0 \le z_1^{-L} \le 0.625$ and $0 \le z_2^{-L} \le 0.54$.

This MOLFP problem is equivalent to the following LP problem can be written as:

P_4':

$\text{Max}\left\{\omega_1\big(1.5\,x_1 + 2.5x_2 - 0.625(2x_1 + 3x_2 + 5)\big) + \quad \omega_2\big(\,2x_1 + 3x_2 - 0.54(5\,x_1 + 5x_2 + 3)\big)\right\}$

Subject to :

$0.31 \le \frac{1.5x_1+2.5x_2}{2x_1+3x_2+5} \le 1.46$,

$0.29 \le \frac{2x_1+3x_2}{5x_1+5x_2+3} \le 2.08$,

$x_1 + x_2 \le 5 \,,\; 3x_1 + x_2 \le 10$,

$2x_1 + x_2 \le 7, \quad x_1 \le 3 , \quad x_1 , x_2 \ge 0$

For $\omega_1 = \omega_2 = 0.5$

The optimal solution of the LP problem P_4' is obtained as: ${x_1}^* = 0 \;,\; {x_2}^* = 5$

The efficient solution of MOLFP problem $P_4$are: ${x_1}^* = 0 \;,\; {x_2}^* = 5,$ with objective value $z_1^{-L} = 0.625 \;,\; z_2^{-L} = 0.54$.

The rough efficient solution of original MORLFP problem is ${x_1}^* = 0 \;, {x_2}^* = 5$ with the rough objective value

$z_1^R = \big[[0.625 \,,\; 1.46] \;:\; [0.31 \,,\; 3.33]\big]$,

$z_2^R = \big[[0.54 \,,\; 2.08] \;:\; [0.29 \,,\; 5]\big]$.

6. Conclusion

A new approach is proposed for solving multiobjective linear fractional programming problems with rough coefficients (MORLFP) problem. For treating the problems use the method of Effati and Pakdaman to convert the MORLFP problem into four multi objective linear fractional programming MOLFP problems. By the method of Dinkelbach, the MOLFP problems is convert to linear programming LP problems . An algorithm is established for characterizing the solutions concept of MORLFP problems .

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الملخص العربي :

في هذه الورقة تناولنا طريقة جديده لحل المشاكل الامثليه الخطيه الكسريه متعددة الاهداف حيث تكون معاملات دوال الهدف Rough intervals تتلخص الطريقة في تحويل المشكلة الامثلية المعطاه الي اربعة مشاكل امثليه خطيه كسرية متعدده الاهداف في صوره ابسط حيث تكون معاملات دوال الهدف اعداد حقيقيه . استعرضنا بعض التعريفات والنظريات واقترحنا خوارزمية لايجاد الحل الامثل للمشكله واعطينا مثال عددي من اجل التوضيح.

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Mathematics

Relative convex body in Euclidean and hyperbolic spaces

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Abstract: This paper is to generalize the concept of convex body to the so called relative convex body in Euclidean space E^n. Some geometrical and topological properties for this kind of sets are discussed. Some properties of the central projection map(Beltrami map) introduced to discuss these concepts in the hyperbolic space H^n..

Key words: Relative convexity, relative open (closed) sets, relative convex body and relative convex surface.

Introduction:

The concept of convex bodies have an important role in differential geometry and represent a very interesting fruitful area of research.

In the last years a lot of mathematicians generalized convexity notion in Euclidean space E^n . Such as K-convexity[4], D-starshaped sets[3], Invexity[2], and Relative convexity[5].

A new kind of generalized convex body for sets in Euclidean and hyperbolic spaces is presented, this kind is called relative convex body. Also some geometrical and topological properties for this kind are discussed. Before this discussion let us survey some definitions and results that help us in this work.

Definition(1).

A subset B of the Euclidean space $E^n$is an open set if it consists entirely of interior points $in(B)$, hence if $B = in(B)$. A subset $B \subset E^n$ is a closed set if it contains its boundary $Bd(B)$, hence if $B \supset Bd(B)$.[7]

Theorem(2).

(1)The Euclidean space E^n and the empty set are open (closed) sets.

(2)The union of any number (finite number)of open (closed) sets is an open (closed) set.

(3)The intersection of a finite number (any number) of open (closed) sets is an open (closed) set.

Theorem(3).

The following statments are equivalent

(1)B is a closed set, that is $B \supset Bd(B)$.

(2)The limit points to $B, Lp(B)$, belong to B, that is $B \supset Lp(B)$.

(3)If the neighbourhood $N(p,\delta) \cap B \neq \emptyset, \forall \delta \geq 0$, then $p \in B$.

(4)The complement of B is an open set.

(5)B is its own closure,$Cl(B)$, that is $B = Cl(B)$.

Where$N(p,\delta) = x \in B: d(p,x) \leq \delta$.[7]

Theorem(4).

For any set B, the following statments are hold:

(1)The interior $In(B)$ and the exterior $Ex(B)$ are open sets, hence $In[In(B)] = In(B)$.

(2)The closure $Cl(B)$ is a closed set, hence $Cl[Cl(B)] = Cl(B)$.

(3)The boundary $Bd(B)$ is a closed, hence $Bd[Bd(B)] \subset Bd(B)$.

(4)The derived set $Lp(B)$ is a closed set, hence $Lp[Lp(B)] \subset Lp(B)$.[7]

Definition(5).
A subset $S \subset E^n$ is a convex set if for each pair of points $x, y \in S$ it is true that the closed line segment $[xy]$ joining x and y lies wholly in S.[1,8]

Definition(6).
Let B be a subset of the Euclidean space E^n and A be a subset of B. The set A is a relatively convex set with respect to B if for each pair of points $p, q \in A$ the closed segment $[pq]$ joining p and q lies wholly in B.[5]

In the following some results are introduced as given in [5].

(1)If A_1 and A_2 are relatively convex with respect to B, then $A_1 \cap A_2$ relatively convex with respect to B. On the other hand for $A_1 \cup A_2$ the above result is no longer valid.

(2)Every subset $A \subset E^n$ is relatively convex with respect to any of its supersets.

(3)Every subset A is relatively convex with respect to its convex hull. Moreover, each subset is relatively convex with respect to any convex superset.

(4)If A is a relatively convex set with respect to B, then every subset of A is relatively convex with respect to B.

(5)Let $A \subset E^n$ be a subset. If every subset $B \subset A$ is relatively convex with respect to A, then A is convex.

4-Relative closed and relative open sets
In the following section, we shall introduce some definitions and results on relative closed and relative open sets.

Definition(7).
Let B be a subset of the Euclidean space E^n and A be a subset of B. The set A is said to be a relatively closed with respect to B if every limit point of A belongs to B.

We denote to the set of all limit points of the set A by A'. Which is called the derived set.[7]

Proposition(8).
The empty set is relative closed set with respect to any set A.

Proof
Let the empty set $\emptyset$ be not relative closed with respect to A. Then, there exists a limit point of $\emptyset$, say x, such that x dose not belong to A. Since the empty set has no limit points, Then this is a contradiction. Therefore, the empty set is relative closed with respect to any set A.

Corollary(9).
The Euclidean space is relative closed with respect to itself.

Proposition(10).
If A_1 and A_2 are relative closed with respect to B, then $A_1 \cap A_2$ and $A_1 \cup A_2$ are also relative closed with respect to B.

Proof
Firstly, since A_1 and A_2 are relative closed with respect to B, then $A_1, A_2 \subset B$ and $A_1', A_2' \subset B$, Thus $(A_1 \cap A_2) \subset B$. Let x be a limit point of $A_1 \cap A_2$ then, $x \in (A_1 \cap A_2)' \subseteq (A_1' \cap A_2')$, this implies that $x \in A_1' \wedge x \in A_2'$, then $x \in B$. Hence $A_1 \cap A_2$ is relative closed with respect to B. Secondly, simillarly we have $(A_1 \cup A_2) \subset B$ and $A_1' \cup A_2' \subset B$, if x is a limit point of $(A_1 \cup A_2)$, then $x \in (A_1 \cup A_2)' = (A_1' \cup A_2')$. Therefore $x \in A_1'$ or $x \in A_2'$. Therefore $x \in A_1'$ or $x \in A_2'$. Therefore, $x \in B$. Hence, $A_1 \cup A_2$ is relative closed with respect to B.

Proposition(11).
The set $A$is relative closed set with respect to itself if and only if A is closed.

Proof
(1)If A is closed, then A contains all limit points of A. Hence A is relative closed with respect to A.
(2)If A is relative closed with respect to itself, then A contains all limit points of A. Thus, A is closed.

Proposition(12).

If A is relative closed with respect to B, then the closure of A is relative closed with respect to the closure of B.

Proof

Since A is relative closed with respect to B, then we have $A \subset B$ and $A' \subset B$. This implies that $A \subset Cl(A) \subset Cl(B)$. Since $Cl(A)$ is closed, then $\left(Cl(A)\right)' \subset Cl(A) \subset Cl(B)$, hence $Cl(A)$ is relatively closed with respect $Cl(B)$.

Remark(13).

(1) Every closed set A is relative closed with respect to any of its supersets.
(2)Every subset is relative closed with respect to any closed superset.

Corollary(14).

Every set A is relative closed with respect to $Cl(A)$.

Proposition(15).

If A is a relatively closed with respect to B, then every subset of A is relatively closed with respect to B.

Proof

Let x be any limit point of $C \subset A \subset B$, then for all open set G containing x, we have $(G - x) \cap C \neq \emptyset$ implies $(G - x) \cap A \neq \emptyset$. This means that x is a limit point for A. Since A is relative closed with respect to B. Hence, x belongs to B. Therefore, C is relative closed with respect to B.

Corollary(16).

If A is relative closed set with respect to B, then the interior of A is relative closed with respect to B.

Lemma(17).

If A is relative closed set with respect to B and C, then A is relative closed with respect to both $B \cap C$ and $B \cup C$.

Proof

Since A is relative closed with respect to B and C, then we have

$A \subset B, A' \subset B$ (1). And

$$A \subset C, A' \subset C \ (2).$$

From (1) and (2), we have $A \subset (B \cap C)$ and $A' \subset (B \cap C)$. Hence, A is relative closed with respect to $B \cap C$. Simillarly, it is easy to see that A is relative closed with respect to $B \cup C$.

Theorem(18).

If A and B are two relative closed sets with respect to C, then $A \cap B$ and $A \cup B$ are also relative closed sets with respect to C.

Proof

Since A and B are relative closed with respect to C. Then

$A \subset C, A' \subset C$(1). And

$B \subset C, B' \subset C$(2).

From (1) and (2), we have $(A \cap B) \subset C$ and$(A' \cap B') \subset C$. Therefore $(A \cap B)' \subset C$. Hence $(A \cap B)$is relative closed set with respect to C.

Simillarly, it is easy to see that $(A \cup B)$ is relative closed with respect to C.

Proposition(19).

If A is relative closed set with respect to both B and C such that $B \not\subset C$. Then, there exists a proper subset $W \subset B$ such that A is relative closed set with respect to W.

Proof

(1)If $C \subset B$, we put$W = C$. Hence, A is relative closed with respect to W.

(2)If $C \not\subset B$ and $B \not\subset C$ (given), then we have $(B \cap C) \subset B$. Let $W = (B \cap C)$, W is a proper subset of B. Since A is relative closed with respect to both B and C. From the above lemma, we see that A is relative closed with respect to W.

Proposition(20).

Let A be a subset of the Euclidean space E^n. If every subset of A is relative closed with respect to A, then A is closed.

Proof

If $B \subset A$ is a relative closed with respect to A, then A contains all the limit points of the subset B. This is also true when we take $B = A$,i.e. A contains all the limit points of A. Therefore, A is closed.

Definition(21).
Let $B \subset E^n$ and $A \subset B$. The set A is called relative open set with respect to B if for all $x \in A$, there exists an open set G containing x such that $x \in G \subset B$, i.e; G is entirely in B.

Remark(22).
(1)The empty set is relative open with respect to any set.

(2)The Euclidean space E^n is relative open with respect to E^n.

(3)The interior of a set A is relative open with respect to A.

(4)Let $A \subset E^n$ be a subset. If every subset $B \subset A$ is relative open with respect to A. Then A is open.

Proposition(23).
The set A is relative open with respect to A if and only if A is open.

Proof
(1)If A is open, then for all $x \in A$, there exists an open set G containing x, such that $x \in G \subset A$. Thus A is relative open with respect to A.

(2)If A is relative open with respect to itself, then for all $x \in A$, there exists an open set G containing x, such that $x \in G \subset A$. This means that A is open.

Theorem(24).
If the two sets A_1 and A_2 are relative open with respect to B, then $A_1 \cap A_2$ and $A_1 \cup A_2$ are relative open with respect to B.

Proof
(1)Since A_1 and A_2 are relative open with respect to B, then we have $A_1 \subset B$ and $A_2 \subset B$, thus $(A_1 \cap A_2) \subset B$. If $x \in (A_1 \cap A_2)$, then $x \in A_1 \wedge x \in A_2$. Again since A_1 and A_2 are relative open with respect to B, then there exist the open sets G_1 and G_2 such that $x \in G_1 \subset B$ and $x \in G_2 \subset B$. Thus $x \in (G_1 \cap G_2) \subset B$. Therefore $A_1 \cap A_2$ is relative open with respect to B.

(2)Since $A_1 \subset B$ and $A_2 \subset B$, then $(A_1 \cup A_2) \subset B$. Let $x \in (A_1 \cup A_2)$, then $x \in A_1$ or $x \in A_2$. Since A_1 and A_2 are relative open with respect to B, then we discuss the following cases:

(a)If $x \in A_1$, then there exists an open set G_1 containing x such that $x \in G_1 \subset B$, then $A_1 \cup A_2$ is relative open with respect to B.

(b)If $x \in A_2$, then there exists an open set G_2 containing x such that $x \in G_2 \subset B$, then $A_1 \cup A_2$ is relative open with respect to B.

(3)If $(A_1 \cap A_2) \neq \emptyset$. In this case, we see that $\emptyset$ is relative open with respect to B.

Proposition(25).
If A and B are two relative open sets with respect to C, then $(A \cap B)$ and $(A \cup B)$ are relative open with respect to C.

Proof
For all $x \in (A \cap B)$, we have $x \in A \wedge x \in B$. Since A and B are relative open with respect to C, then there exist the open sets G_1 and G_2 such that $x \in G_1 \subset C$ and $x \in G_2 \subset C$. Hence, $x \in (G_1 \cap G_2) \subset C$. Therefore, $A \cap B$ is relative open with respect to C. Simillary, it is easy to see that $A \cup B$ is relative open with respect to C.

5-Relative convex body
In this section, we define relative convex body and study some geometrical properties for this concept.

Definition(26).
Let B subset of E^n, and $A \subset B$. If the subset A is bounded and relative closed with respect to B, then A is called relative compact with respect to B.

Remark(27).
(1)If A is relative compact set with respect to B, then every subset of A is relative compact set with respect to B.

(2)If A is relative compact set with respect to B and C, then A is relative compact set with respect to both $B \cap C$ and $B \cup C$.

Definition(28).

Let $A \subset B$, A is called relative convex body with respect to B if the following conditions are satisfied:

(1)A is relative compact with respect to B.

(2)A is relative convex with respect to B.

(3)A has non-empty interior.

Definition(29).

The relative convex surface is the boundary of the relative convex body.

It is easy to see that

(1)if the two sets A_1 and A_2 are relative convex bodies with respect to B. Then, we have

i)$(A_1 \cap A_2) \neq \emptyset$ is relative convex body with respect to B. (

ii)$A_1 \cup A_2$ is not relative convex body in general.(

(2)Every non-empty subset $A \subset E^n$ is relatively convex body with respect to E^n.

(3)Every convex body is relatively convex body with respect to any of its supersets.

(4)If A is a relatively convex body with

respect to B, then every non-empty bounded subset of A is relatively convex body with respect to B.

Proposition(30).

If A is a relative convex body with respect to both B and C, such that B is not a subset of C. Then there exists a proper subset $W \subset B$ such that A is relative convex body with respect to W.

Proof

(1)Assume that $C \subset B$ and put $W = C$. Hence A is relative convex body with respect to W.

(2)Assume that $C \not\subset B$ and($B \not\subset C$,given), let $W = (B \cap C)$, clearly W is a proper subset of B. Since A is relative convex body with respect to both B and C. Therefore, we have A is relative convex body with respect to W.

6-Relative convex in hyperbolic space

Now we devote our study to the concept of relative convex in hyperbolic space H^n. The most convenient model of the n-dimensional hyperbolic space for the present work is the spherical one H^n which might be defined as follows [8],[5]:for

$$H^n = (x^1, x^2, \ldots\ldots, x^n) \in V^{n+1} : -(-(x^1)^2 + \sum_{i=2}^{n+1} (x^i)^2) = 1 \quad , x^1 \succ 0$$

and also in the metric, where V^{n+1} denotes the Minkowski space $(R^{n+1}, <,>)$ with the pseudo-Riemannian metric $<,> = -dx^1 \odot dx^1 + \sum dx^i \odot dx^i$. The metric when restricted to H^n yields a Riemannian metric with constant sectional curvature $K = -1$[5]. As H^n is a complete simply connected Riemannian manifold with negative sectional curvature, then each pair of points $p, q \in H^n$ are joined with a unique geodesic segment [8]. Therefore, H^n is starshaped. The Beltremi(or central projection)map $\beta: H^n \to E^n$ is defined to be the map which takes $x \in H^n$ to the intersection of the Euclidean space $x^1 = 1$ with the straight line through x and the origin 0 of V^{n+1}. The map β takes the whole of H^n diffeomorphically to the open ball $B(p, l)$ of radius 1 and center at $p = (1,0,0, \ldots ,0)$. Furthermore, the map β is a geodesic map and so K-totally geodesic submanifolds of H^n are mapped under β onto k-planes in E^n. We can also show that closed, open, compact, bounded and starshaped subsets of H^n are mapped under β to subsets of $B(p, l)$ of the same type. It worth mentioning that the inverse of map β has the same properties of β.[6]

Lemma(31).

The central projection map preserves limit points of sets.

Proof

Obvious.

Proposition(32).

The central projection map preserves relative closed property of set.

Proof

Let B be a subset of H^n and A is a relative closed set with respect to B. Then, for any limit point of A, say x, belongs to B. If we apply the central projection map β, then we have $\beta(x)$ is a limit point of $\beta(A)$ and $\beta(A) \subset \beta(B)$. Since $x \in B$, then $\beta(x) \in \beta(B)$. Therefore, $\beta(x)$ is a limit point of $\beta(A) \subset \beta(B)$ and $\beta(x) \in \beta(B)$. Hence, $\beta(A)$ is relative closed set with respect to $\beta(B)$.

Proposition(33).

The central projection map preserves on relative open sets.

Proof

Let B be a subset in H^n such that is relative open set with respect to B. Then, for any $x \in B$ and there exists an open set G containing x such that $x \in G \subset B$. If we apply β then $\beta(x) \in G' \subset \beta(B)$. Therefore the centeral projection map preserves on relative open sets, where $\beta(G) = G'$ is open set.

Proposition(34).

The central projection preserves on relative convexity.

Proof

Let B be a closed connected set in H^n and A be a subset of B. Assume that x, y in A and the closed geodesic segment ,say αxy, which is determined by x and y is in B. Then if we apply β and assume that the closed segment $[\beta(x)\beta(y)]$ is not in $\beta(B)$, this means that there exists at least one point belongs to $[\beta(x)\beta(y)]$ but not belongs to $\beta(B)$, which is a contradiction with the fact that β preserves on the interior, exterior and boundary points of B. Hence $\beta(x), \beta(y)$ are in $\beta(A)$ and $[\beta(x)\beta(y)]$ is in $\beta(B)$. This implies that $\beta(A)$ is relative convex with respect to $\beta(B)$.

Corollary(35).

the central projection map preserves on the concept of relative convex body.

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تحدب النسبي للأجسام في الفراغ النوني والزائدي

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ان تحدب الأجسام يمثل دور هام في الهندسة التفاضلية ويمثل مساحة واسعة للبحث حيث ان في السنوات الأخيرة كثير من علماء الرياضيات قاموا بتعميم فرض التحدب في الفراغ النوني, في هذا البحث قمنا بتعميم تحدب الأجسام بما يسمي تحدب الأجسام النسبي وقمنا بدراسة بعض الخواص التوبولوجية والهندسية لهذا النوع من التحدب, ايضا في هذا البحث عرفنا انواع جديدة من الخواص الهندسية والتوبولوجية للتحدب النسبي للاجسام في الفراغ الزائدي وحصلنا علي ترابط بين الفراغات النوني والزائدي من خلال استخدام راسم بيلترامي.

Physics