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NODAL LINE FINITE DIFFERENCE METHOD FOR THE ANALYSIS OF CIRCULAR CYLINDRICAL SHELLS

تحليل المنشآت القشرية الاسطوانية بطريقة الفروق المحددة لخطوط التقسيم

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الخلاصة ـ يساهم هذا البحث في ترسيخ وتوسيع مجال تطبيق طريقة الباحث المسماه بالفروق المحددة لخطوط التقسيم . ويتناول هذا البحث تحليل المنشآت القشرية الاسطوانية المرتكزة ارتكازا بسيطا على حافتين متقابلتين في اتجاهها الطولي. ولتطبيق هذه الطريقة تم تقسيم القشرة الاسطوانية الى مجموعة من الخطوط المتوازية وتم التعبير عن مركبات الازاحة الثلاث على خطوط التقسيم باستخدام المعادلات المثلثية التي لها خاصية عدم الازدواج . وفي هذا البحث تم اشتقاق المعادلات التفاضلية الجزئية الاتية الثلاث التي تحكم سلوك القشرة الاسطوانية بدقة دون اللجوء الى اى تبسيط او فروض من شأنها الاخلال بعموميتها . لذا فان معادلات الفروق التي تم اشتقاقها في هذا البحث صالحة لتحليل المنشآت القشرية الاسطوانية باختلاف اشكالها الهندسية المعرضة لاى نوع من انواع الاحمال الاستاتيكية . هذا بالاضافة الى ان عناصر مصفوفة معادلات الفروق تنحصر حول قطرها الامر الذي يجعل من السهل اعادة ترتيبها في مصفوفة مستطيلة ذات عرض بسيط مما يوفر وبدرجة كبيرة سعة التخزين والوقت اللازم للحل . ولقد تم مقارنة نتائج الطريقة المذكورة في هذا البحث بنتائج الحل التحليلي واظهرت المقارنة دقة وكفاءة هذه الطريقة

ABSTRACT :*In the present work, analysis of simply supported thin elastic circular cylindrical shells via the Author's nodal line finite difference method is presented. The analysis starts with the accurate derivation of the three governing partial differential equations without any simplifications or assumptions that affect the generality of these equations. The method requires the division of the shell into a mesh of parallel fictitious nodal lines in the longitudinal direction. Simple trigonometric basic functions having uncoupling property are used to express the displacement components variation along the nodal lines. Consequently, three ordinary differential equations are derived and cast into three simultaneous nodal line difference equations. The final matrix of these difference equations is a square banded matrix with small half band width which reduces, to a great extent, the core storage and the time of computations. The results obtained are compared with those from analytical solutions and the comparison demonstrated a close agreement.*

INTRODUCTION

Circular cylindrical shells belong to the class of stressed-skin structures which by virtue of their geometry and small flexural rigidity tend to carry applied loads by direct stresses lying in their plane accompanied by small bending moments. The formulation of the basic equations of thin elastic shells, in particular, circular cylindrical shells, due to their importance for practical applications, has been the subject of considerable research interest which received repeated attention in several publications. The variety of the resulting equations found in the literature, although all of them are based on the same basic Kirchhoff's hypotheses, are due to variation in rigor in their derivations and modified approximations in the subsequent formulations.

A common difficulty with the rigorous basic equations for circular cylindrical shell such as Flugge's, Timoshenko's, Vlasov's and Goldenveizer's [8,11,13,14] is that their general solutions faced with mathematical complexities. For this reason many attempts have been made throughout the history of the shell theory to simplify the basic equations. A number of basic equations of a simplified nature that are applicable to circular cylindrical shells, having special geometrical relations (long - short - shallow), or deformations conditions (inextensible) have been suggested by many authors such as Donnell, Gibson, Schorer, Finsterwalder and others. Detailed discussion of these simplified basic equations can be found in many references such as [10,11,12,15]

Herein the attention is directed to the application of the nodal line finite difference method **NLFDM** for the analysis of simply supported thin elastic circular cylindrical shells. This method is a new semi-analytical approach, developed earlier by the Author, which has been successfully applied to the bending and in-plane stress analysis of rectangular and circular plates [1,2,3,4,5,6,7]. The nodal line finite difference method **NLFDM** can be briefly considered as a solution technique which transforms the partial differential equations into algebraic difference equations to be applied at nodal lines on the actual structure. This technique calls for the use of analytic series in one direction while in the other direction the differential operators are replaced by simple difference expressions.

The present analysis starts with the rigorous derivation of the three partial differential equations governing the displacement components of thin elastic circular cylindrical shells. The derivation is only based on the well known Kirchhoff's assumptions, used in all bending theories, without any other assumptions or simplifications that affect the generality of the governing partial differential equations. The application of the proposed technique requires the division of the shell into a mesh of parallel fictitious nodal lines in the longitudinal direction. Simple trigonometric basic functions are chosen to express the displacement components variation along the nodal lines. These basic functions should satisfy a priori the boundary conditions at the two opposite end diaphragms. The three governing partial differential equations are then transformed into three simultaneous ordinary differential equations which are in turn transformed into three nodal line difference equations by means of replacing the derivatives by difference expressions. The application of these nodal line difference equations at each nodal line results in a system of linear algebraic equations. The final square matrix of these equations is a symmetrical banded matrix with small half band width which reduces drastically the storage requirements and the computation time. The present formulation has the advantage of being general and applicable to simply supported thin circular cylindrical shells having any geometrical relations under any type of loading conditions. Numerical examples are included to demonstrate the applicability and the accuracy of the proposed method.

METHOD OF ANALYSIS

A circular cylindrical shell may be thought of as a surface generated by a straight line, known as the generator, moving over a plane arc of a circle. The present study deals with the stress analysis of thin elastic isotropic cylindrical shells with relatively small thickness compared with each other dimensions and compared with its arc radius. The analysis starts with establishing equilibrium of a differential element cut out from the shell, Fig. 1-b, by two adjacent generators and two cross sections perpendicular to the x-axis, and its position is defined by the coordinate x and the angle ϕ. The forces acting on the sides of the element are shown in Fig. 1-b. The load intensity components X, Y and Z are considered to be distributed over the surface of the element.

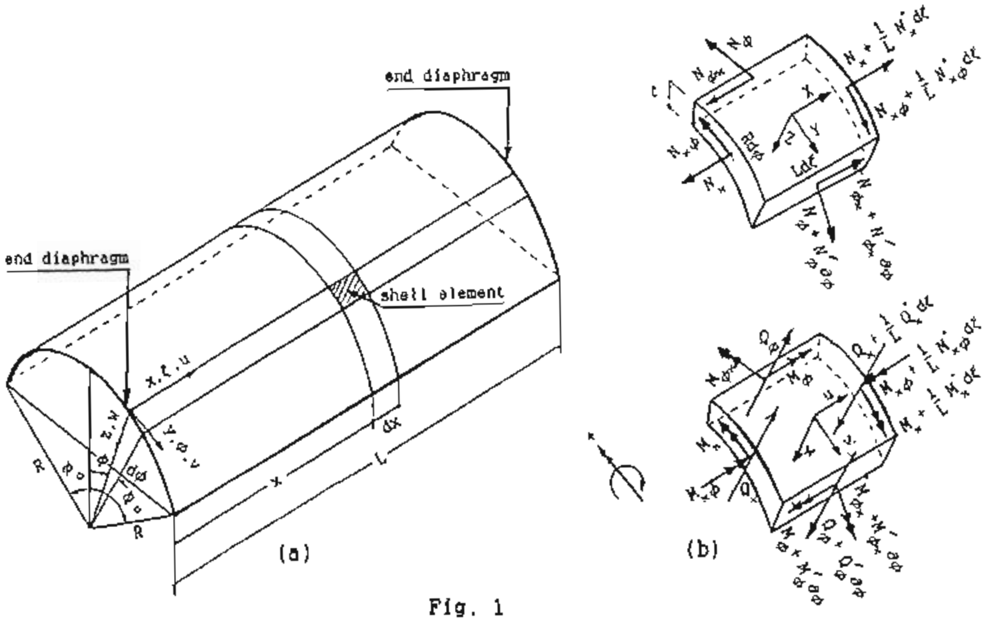

Fig. 1

Considering the equilibrium of the element and summing up the forces in the x, ϕ and z-directions, six equilibrium equations are obtained. One of these equations, derived from the equilibrium conditions of bending moments in the z-direction, is self satisfied. The other five equations are reduced to the following three partial differential equations.

$$\left.\begin{aligned}
&\lambda N_x^{\cdot} + N_{\phi x}' + R X = 0\\
&\lambda N_{x\phi}^{\cdot} + N_{\phi}' - \frac{1}{L} M_{x\phi}^{\cdot} - \frac{1}{R} M_{\phi}' + R Y = 0\\
&N_{\phi} + \frac{\lambda}{L} M_x^{\cdot\cdot} + \frac{1}{L} (M_{x\phi}^{\cdot\prime} + M_{\phi x}^{\cdot\prime}) + \frac{1}{R} M_{\phi}'' + R Z = 0
\end{aligned}\right\} \qquad (1)$$

where $(\;)^{\cdot} = \frac{\partial}{\partial \xi}$, $(\;)' = \frac{\partial}{\partial \phi}$, $\xi = \frac{x}{L}$ and $\lambda = \frac{R}{L}$

The displacement at any point on the middle surface of the shell can be resolved into three components u, v and w in the x-direction, in the direction of the tangent at that point and in the direction of the inward directed normal, respictively. The expressions for strains and curvatures at any point of the middle surface of the shell may be derived as follows

$$\left.\begin{aligned}
&\epsilon_x = \frac{1}{R} \lambda u^{\cdot} , && \chi_x = \frac{\lambda^2}{R^2} w^{\cdot\cdot}\\
&\epsilon_{\phi} = \frac{1}{R} (v' - w) , && \chi_{\phi} = \frac{1}{R^2} (v' + w'')\\
&\epsilon_{x\phi} = \frac{1}{R} (u' + \lambda v^{\cdot}) , && \chi_{x\phi} = \frac{\lambda}{R^2} (v^{\cdot} + w^{\cdot\prime})\\
&\epsilon_{\phi x} = \frac{1}{R} (u' + \lambda v^{\cdot}) , && \chi_{\phi x} = \frac{\lambda}{R^2} w^{\cdot\prime}
\end{aligned}\right\} \qquad (2)$$

Essentially all the analysis of thin shells is based on the Kirchhoff's assumptions which state that the material of the shell is linearly elastic, isotropic, homogeneous and obeys Hook's law. In addition to these assumptions, all displacements of the shell are assumed to be small compared to the shell thickness. Taking these assumptions into considerations, strains as well as stresses at any point on the surface at a small distance , z <± $t/2$, from the middle surface may be obtained. Stress resultants and stress couples at any point of the shell would be obtained by the integration of stresses over the thickness. Detailed derivation of this preceeding procedure are not included herein, but the final experissions relating the internal forces directly to the displacement components are given as follows

$$\left.\begin{aligned}
N_x &= \frac{D}{R}\left[\lambda u^{\cdot} + \nu v' + k\lambda^2 w^{\cdot\cdot} - \nu w\right] \\
N_\phi &= \frac{D}{R}\left[\nu\lambda u^{\cdot} + v' - k w'' - (1+k) w\right] \\
N_{x\phi} &= \frac{D}{R}\,\frac{1-\nu}{2}\left[u' + (1+k)\lambda v^{\cdot} + k\lambda w^{\cdot\prime}\right] \\
N_{\phi x} &= \frac{D}{R}\,\frac{1-\nu}{2}\left[(1+k) u' + \lambda v^{\cdot} - k\lambda w^{\cdot\prime}\right] \\
M_x &= -\frac{B}{R^2}\left[\lambda u^{\cdot} + \nu v' + \lambda^2 w^{\cdot\cdot} + \nu w''\right] \\
M_\phi &= -\frac{B}{R^2}\left[\nu\lambda^2 w^{\cdot\cdot} + w'' + w\right] \\
M_{x\phi} &= -\frac{B}{R^2}\,\frac{1-\nu}{2}\left[2\lambda v^{\cdot} + 2\lambda w^{\cdot\prime}\right] \\
M_{\phi x} &= -\frac{B}{R^2}\,\frac{1-\nu}{2}\left[-u' + \lambda v^{\cdot} + 2\lambda w^{\cdot\prime}\right] \\
Q_x &= -\frac{B}{R^3}\left[\left(\lambda^2 u^{\cdot\cdot} - \frac{1-\nu}{2} u''\right) + \frac{1+\nu}{2}\lambda v^{\cdot\prime} + \lambda^3 w^{\cdot\cdot\cdot} + \lambda w^{\cdot\prime\prime}\right] \\
Q_\phi &= -\frac{B}{R^3}\left[(1-\nu)\lambda^2 v^{\cdot\cdot} + \lambda^2 w^{\cdot\cdot\prime} + w''' + w'\right]
\end{aligned}\right\} \quad (3)$$

where $k = \dfrac{t^2}{12R^2}$, $D = \dfrac{Et}{1-\nu^2} = \dfrac{12}{t^2} B$, $B = k R^2 D$,

D, B are the in-plane and bending stiffness of the shell

Upon substitution of the internal forces from equation (3) into equations (1), three simultaneous partial differential equations are obtained as follows

$$\left.\begin{aligned}
&\left[\lambda^2 u^{\cdot\cdot} + \frac{1-\nu}{2}(1+k) u''\right] + \left[\frac{1+\nu}{2}\lambda v^{\cdot\prime}\right] - \left[\nu\lambda w^{\cdot} - k\lambda^3 w^{\cdot\cdot\cdot} + \frac{1-\nu}{2} k\lambda w^{\cdot\prime\prime}\right] = -\frac{R^2}{D} X \\
&\left[\frac{1+\nu}{2}\lambda u^{\cdot\prime}\right] + \left[v'' + \frac{1-\nu}{2}(1+3k)\lambda^2 v^{\cdot\cdot}\right] - \left[w' - \frac{3-\nu}{2} k\lambda^2 w^{\cdot\cdot\prime}\right] = -\frac{R^2}{D} Y \\
&\left[\nu\lambda u^{\cdot} - k\lambda^3 u^{\cdot\cdot\cdot} + \frac{1-\nu}{2} k\lambda u^{\cdot\prime\prime}\right] + \left[v' - \frac{3-\nu}{2} k\lambda^2 v^{\cdot\cdot\prime}\right] - \\
&\qquad \left[(1+k) w + 2k w'' + k\left(\lambda^4 w^{\cdot\cdot\cdot\cdot} + 2\lambda^2 w^{\cdot\cdot\prime\prime} + w''''\right)\right] = -\frac{R^4}{B} k Z = -\frac{R^2}{D} Z
\end{aligned}\right\} \quad (4)$$

Within the scope of the mentioned Kirchhoff's assumptions, used nearly in all bending theories of circular cylindrical shell, there are no other assumptions or simplifications that affect the generality of these partial differential equations. Therefore, these equations describe generally the structural behavior of this type of shell regardless of its geometrical relations. This means that these equations are applicable to all classes of circular cylindrical shells, (short - intermediate - long - open - closed - shallow - deep), under any type of static loading conditions.

a - Nodal Line Difference Equations

The solution technique of the governing partial differential equations (4) is carried out herein by using nodal line finite difference method which requires the division of the shell into parallel fictitious nodal lines in the longitudinal direction. The analysis involves the use of simple trigonometric basic function to express the displacement components variation along these nodal lines. These basic functions should satisfy a priori the boundary conditions at the two opposite ends in the longitudinal directions.

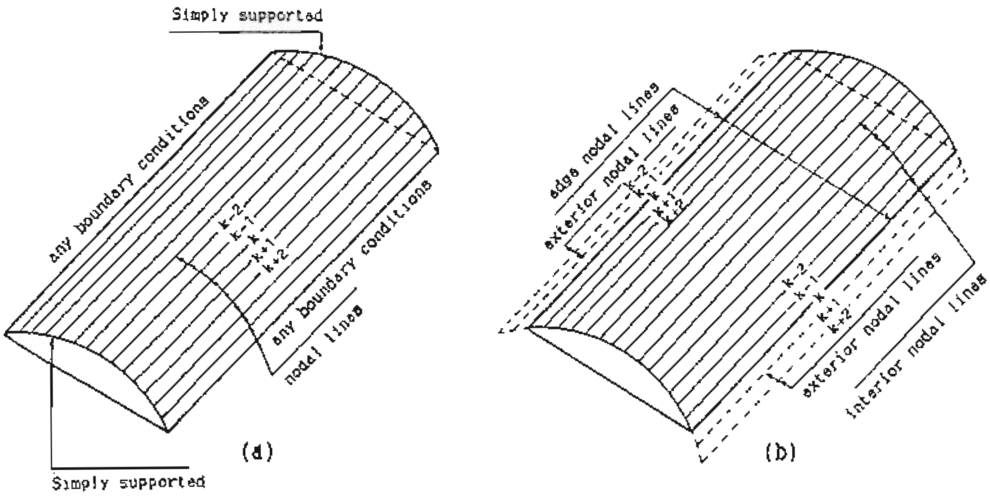

Fig. 2

The displacement components variation can be considered as the summation of the terms of the used basic functions multiplied by a single variable nodal line parameters. It is usually assumed that the shell is simply supported at a two opposite ends in the longitudinal direction termed as end diaphragms. Accordingly, the displacement components variation along the nodal lines are proposed as

$$\left.\begin{aligned} u_k &= \sum_{m=1}^{r} U_{m,k}(\phi) \cos m\pi\xi \\ v_k &= \sum_{m=1}^{r} V_{m,k}(\phi) \sin m\pi\xi \\ w_k &= \sum_{m=1}^{r} W_{m,k}(\phi) \sin m\pi\xi \end{aligned}\right\} \qquad (5)$$

The load intensity components are then resolved into series similar to that used to express the displacement components variation. Therefore the load intensity components can be expressed as

$$\left.\begin{aligned} X_k &= \sum_{m=1}^{r} q^x_{m,k}(\phi) \cos m\pi\xi \\ Y_k &= \sum_{m=1}^{r} q^y_{m,k}(\phi) \sin m\pi\xi \\ Z_k &= \sum_{m=1}^{r} q^z_{m,k}(\phi) \sin m\pi\xi \end{aligned}\right\} \qquad (6)$$

Upon substitution of equations (5) and (6) into equations (4), three ordinary differential equations are then obtained for each term, m, as follows

$$
\left.
\begin{aligned}
&\left[\eta^2 - \frac{1-\nu}{2}(1+k)U''_{m,k}\right] - \left[\frac{1+\nu}{2}\eta V'_{m,k}\right] + \left[\frac{1-\nu}{2}k\eta W''_{m,k} + \eta(\nu+k\eta^2)W_{m,k}\right] = \frac{R^2}{D}q^x_{m,k} \\
&\left[\frac{1+\nu}{2}\eta U'_{m,k}\right] - \left[V''_{m,k} - \frac{1-\nu}{2}(1+3k)\eta^2 V_{m,k}\right] + \left[(1+\frac{3-\nu}{2}k\eta^2)W'_{m,k}\right] = \frac{R^2}{D}q^y_{m,k} \\
&\left[\frac{1-\nu}{2}k\eta U''_{m,k} + \eta(\nu+k\eta^2)U_{m,k}\right] - \left[(1 + \frac{3-\nu}{2}k\eta^2)V'_{m,k}\right] + \\
&\qquad \left[kW''''_{m,k} - 2k(\eta^2-1)W''_{m,k} + (1+k+k\eta^4)W_{m,k}\right] = \frac{R^2}{D}q^z_{m,k}
\end{aligned}
\right\} \quad (7)
$$

where $\eta = \lambda\, m\, \pi$

Using the central finite difference technique in the circumferential direction, the above ordinary differential equations are cast into the following three simultaneous nodal line difference equations as

$$
\left.
\begin{aligned}
&[0\ 0\ 0\ |-C^1_m\ \ C^2_m\ \ C^3_m|C^4_m\ \ 0\ \ C^5_m\ |-C^1_m\ -C^2_m\ \ C^3_m|0\ 0\ 0\ ]\{\delta_m\} = \frac{R^2}{D}\Delta^2 q^x_{m,k} \\
&[0\ 0\ 0\ |-C^2_m\ \ -1\ -C^6_m|0\ \ C^7_m\ \ 0\ |\ C^2_m\ -1\ \ C^6_m|0\ 0\ 0\ ]\{\delta_m\} = \frac{R^2}{D}\Delta^2 q^y_{m,k} \\
&[0\ 0\ C^8_m|\ C^3_m\ \ C^6_m\ -C^9_m|C^5_m\ \ 0\ \ C^{10}_m|\ C^3_m\ -C^6_m\ -C^9_m|0\ 0\ C^8_m]\{\delta_m\} = \frac{R^2}{D}\Delta^2 q^z_{m,k}
\end{aligned}
\right\} \quad (8)
$$

where

$C^1_m = \frac{1-\nu}{2}(1+k)$, $C^2_m = \frac{1+\nu}{4}\eta\Delta$. $C^3_m = \frac{1-\nu}{2}k\eta$, $C^4_m = 2C^1_m + \eta^2\Delta^2$,

$C^5_m = \eta(\nu + k\eta^2)\Delta^2 - 2C^3_m$, $C^6_m = \frac{1}{2}(1+ \frac{3-\nu}{2}k\eta^2)\Delta$, $C^7_m = 2 + \frac{1-\nu}{2}(1+3k)\eta^2\Delta^2$,

$C^8_m = k\alpha^2$, $C^9_m = 4C^8_m + 2k(\eta^2-1)$, $C^{10}_m = 6C^8_m\alpha^2 + 4k(\eta^2-1) + (1+k+k\eta^4)\Delta^2$,

$\{\delta_m\} = \{\ (\delta_{m,k-2})\ \ (\delta_{m,k-1})\ \ (\delta_{m,k})\ \ (\delta_{m,k+1})\ \ (\delta_{m,k+2})\ \}^T$,

$\{\delta_{m,k-2}\} = \{U_{m,k-2}\ V_{m,k-2}\ W_{m,k-2}\}^T$, $\{\delta_{m,k-1}\} = \{U_{m,k-1}\ V_{m,k-1}\ W_{m,k-1}\}^T$,

$\{\delta_{m,k}\} = \{U_{m,k}\ V_{m,k}\ W_{m,k}\}^T$, $\{\delta_{m,k+1}\} = \{U_{m,k+1}\ V_{m,k+1}\ W_{m,k+1}\}^T$,

$\{\delta_{m,k+2}\} = \{U_{m,k+2}\ V_{m,k+2}\ W_{m,k+2}\}^T$. $\Delta = \Delta\phi$ and $\alpha = \frac{1}{\Delta}$

The application of these nodal line difference equations at each nodal line results in a system of linear algebraic equations. The final matrix of these equations is a square matrix with size 3Nx3N where N is the number of the nodal lines. The matrix is banded with small band width equals to 15, and hence this matrix can be rearranged in a rectangular matrix with size 3Nx17. Accordingly, core storage as well as the time of execution are remarkably reduced. The unknown nodal line parameters for each term of the used basic functions are then obtained from the solution of the above mentioned system of algebraic equations.

b - Internal Forces

Once the unknown nodal line parameters, equations (5), are determined, it is a simple matter to obtain the internal forces along any nodal line of the shell. Upon substitution of equations (5) into equations (3) and by applying the central finite difference technique in the circumferential direction, the internal forces can be expressed in a nodal line difference form as

$$
\left.
\begin{aligned}
N_x &= \frac{D}{2R}\sum_{m=1}^{r}\sin m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} 0 & -C_2^1 & 0 & -C_4^1 & 0 & -C_6^1 & 0 & C_2^1 & 0 \end{array}\right]\{\bar{\delta}_m\} \\
N_\phi &= \frac{D}{2R}\sum_{m=1}^{r}\sin m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} 0 & -C_2^2 & -C_3^2 & -C_4^2 & 0 & C_6^2 & 0 & C_2^2 & -C_3^2 \end{array}\right]\{\bar{\delta}_m\} \\
N_{x\phi} &= \frac{1-\nu}{2}\,\frac{D}{2R}\sum_{m=1}^{r}\cos m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} -C_2^2 & 0 & -C_3^3 & 0 & C_5^3 & 0 & C_2^2 & 0 & C_3^3 \end{array}\right]\{\bar{\delta}_m\} \\
N_{\phi x} &= \frac{1-\nu}{2}\,\frac{D}{2R}\sum_{m=1}^{r}\cos m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} -C_1^4 & 0 & C_3^3 & 0 & C_4^1 & 0 & C_1^4 & 0 & -C_3^3 \end{array}\right]\{\bar{\delta}_m\} \\
M_x &= \frac{B}{2R^2}\sum_{m=1}^{r}\sin m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} 0 & C_2^1 & -C_3^5 & C_4^1 & 0 & C_6^5 & 0 & -C_2^1 & -C_3^5 \end{array}\right]\{\bar{\delta}_m\} \\
M_\phi &= \frac{B}{2R^2}\sum_{m=1}^{r}\sin m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & -C_3^6 & 0 & 0 & C_6^6 & 0 & 0 & -C_3^6 \end{array}\right]\{\bar{\delta}_m\} \\
M_{x\phi} &= \frac{1-\nu}{2}\,\frac{B}{2R^2}\sum_{m=1}^{r}\cos m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & C_3^7 & 0 & -C_5^7 & 0 & 0 & 0 & -C_3^7 \end{array}\right]\{\bar{\delta}_m\} \\
M_{\phi x} &= \frac{1-\nu}{2}\,\frac{B}{2R^2}\sum_{m=1}^{r}\cos m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} -C_2^2 & 0 & C_3^7 & 0 & -C_4^1 & 0 & C_2^2 & 0 & -C_3^7 \end{array}\right]\{\bar{\delta}_m\} \\
Q_x &= \frac{B}{2R^3}\sum_{m=1}^{r}\cos m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} C_1^8 & C_2^8 & -C_3^8 & C_4^8 & 0 & C_6^8 & C_1^8 & -C_2^8 & -C_3^8 \end{array}\right]\{\bar{\delta}_m\} \\
Q_\phi &= \frac{B}{2R^3}\sum_{m=1}^{r}\sin m\pi\xi\ \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & -C_3^9 & 0 & C_5^9 & 0 & 0 & 0 & C_3^9 \end{array}\right]\{\bar{\delta}_m\} \\
&\qquad + \alpha^3\,(W_{m,k-2} - W_{m,k+2})
\end{aligned}
\right\} \qquad (9)
$$

where

$C_2^1 = \nu\alpha$, $C_4^1 = 2\eta$, $C_6^1 = 2(\nu+k\eta^2)$, $C_2^2 = \alpha$, $C_3^2 = 2k\alpha^2$, $C_4^2 = 2\nu\eta$,

$C_6^2 = 4k\alpha^2 - 2(1+k)$, $C_3^3 = k\eta\alpha$, $C_5^3 = 2(1+k)\eta$, $C_1^4 = (1+k)\alpha$, $C_3^5 = 2\nu\alpha^2$,

$C_6^5 = 4\nu\alpha^2 + 2\eta^2$, $C_3^6 = 2\alpha^2$, $C_6^6 = 4\alpha^2 - 2(1-\nu\eta^2)$, $C_3^7 = 2\eta\alpha$, $C_5^7 = 4\eta$,

$C_1^8 = (1-\nu)\alpha^2$, $C_2^8 = \frac{1+\nu}{2}\eta\alpha$, $C_3^8 = 2\eta\alpha^2$, $C_4^8 = 2\eta^2 - 2(1-\nu)\alpha^2$,

$C_6^8 = 2\eta(2\alpha^2+\eta^2)$, $C_3^9 = 2\alpha^3 - (1-\eta^2)\alpha$, $C_5^9 = 2(1-\nu)\eta^2$

and $\{\bar{\delta}_m\} = \{ (\delta_{m,k-1}) \quad (\delta_{m,k}) \quad (\delta_{m,k+1}) \}^T$

c - Boundary Conditions

The solution of a specific two-dimensional problem of bounded continuous domain should satisfy the equilibrium, compatibility and boundary conditions. In the analysis of circular cylindrical shells, the boundary conditions at the shell four edges must be prescribed in advance. The two opposite ends in the longitudinal direction control the choice of the basic functions which express the displacement components variation in the longitudinal direction. The other two opposite edges in the circumferential direction can take any combination of boundary conditions.

The analysis of circular cylindrical shell using the proposed technique requires the application of the nodal line difference equations (8) at any nodal line within the shell including the edge nodal lines. If the pivotal nodal line, k, coincide with the left or the right edge nodal line, Fig. 2-b, two fictitious nodal lines outside the shell will be introduced. According to the prescribed boundary conditions at the edge nodal lines, the exterior nodal line parameters have to be expressed in terms of the edge and the two adjacent interior nodal lines. For each term of the used basic functions, the letf and the right exterior nodal line parameters can be expressed as follows.

1 - Simply supported edge $\{u=0 \;,\; w=0 \;,\; N_{\phi}=0 \;,\; M_{\phi}=0\}$

$$\left.\begin{aligned}
U_{m,k\mp1} &= [-1 \quad 0 \quad 0]\{\delta_{m,k\pm1}\} + [-\psi^1_1 \quad 0 \quad 0]\{P_m\} \\
V_{m,k\mp1} &= [0 \quad 1 \quad 0]\{\delta_{m,k\pm1}\} \\
W_{m,k\mp1} &= [0 \quad 0 \quad -1]\{\delta_{m,k\pm1}\} \\
W_{m,k\mp2} &= [0 \quad 0 \quad -1]\{\delta_{m,k\pm2}\} + [\psi^1_3 \quad 0 \quad \psi^1_2]\{P_m\}
\end{aligned}\right\} \qquad (10)$$

where $\psi^1_1 = 1/C^1_m \;,\; \psi^1_2 = 1/C^8_m \;,\; \psi^1_3 = C^3_m\psi^1_1\psi^1_2$

and $\{P_m\} = \frac{R^2}{D}\Delta^2\{\, q^x_{m,k} \quad q^y_{m,k} \quad q^z_{m,k} \,\}^T$

2 - Clamped edge $\{u=0 \;,\; v=0 \;,\; w=0 \;,\; w'=0\}$

$$\left.\begin{aligned}
U_{m,k\mp1} &= [-\psi^2_1 \quad \mp\psi^2_2 \quad \psi^2_3]\{\delta_{m,k\pm1}\} - \psi^2_0[1 \quad \pm C^2_m \quad 0]\{P_m\} \\
V_{m,k\mp1} &= [\pm\psi^2_4 \quad -\psi^2_1 \quad \mp\psi^2_5]\{\delta_{m,k\pm1}\} \pm \psi^2_0[C^2_m \quad \mp C^1_m \quad 0]\{P_m\} \\
W_{m,k\mp1} &= [0 \quad 0 \quad 1]\{\delta_{m,k\pm1}\} \\
W_{m,k\mp2} &= [\psi^2_6 \quad \pm\psi^2_7 \quad \psi^2_8]\{\delta_{m,k\pm1}\} + [0 \quad 0 \quad -1]\{\delta_{m,k\pm2}\} \\
&\quad + [\pm\psi^2_9 \quad \pm\psi^2_{10} \quad \psi^2_{11}]\{P_m\}
\end{aligned}\right\} \qquad (11)$$

where $\psi^2_0=1/(C^1_m+C^2_mC^2_m)$, $\psi^2_1=(C^1_m-C^2_mC^2_m)\psi^2_0$, $\psi^2_2=2C^2_m\psi^2_0$, $\psi^2_3=2C^3_m\psi^2_0$,
$\psi^2_4=C^1_m\psi^2_2$, $\psi^2_5=C^2_m\psi^2_3$, $\psi^2_6=(C^3_m\psi^2_1-C^3_m-C^6_m\psi^2_4)/C^8_m$,
$\psi^2_7=(C^6_m+C^6_m\psi^2_1+C^9_m\psi^2_2)/C^8_m$, $\psi^2_8=(2C^9_m-C^3_m\psi^2_3+C^6_m\psi^2_5)$,
$\psi^2_9=\psi^2_0(C^3_m-C^6_mC^2_m)/C^8_m$, $\psi^2_{10}=\psi^2_0(C^3_mC^2_m+C^1_mC^6_m)/C^8_m$ and $\psi^2_{11}=1/C^8_m$

3 - Guided edge $\{u'=0 \;,\; v=0 \;,\; w'=0 \;,\; Q_{\phi}=0\}$

$$\left.\begin{aligned}
U_{m,k\mp1} &= [1 \quad 0 \quad 0]\{\delta_{m,k\pm1}\} \\
V_{m,k\mp1} &= [0 \quad -1 \quad 0]\{\delta_{m,k\pm1}\} \\
W_{m,k\mp1} &= [0 \quad 0 \quad 1]\{\delta_{m,k\pm1}\} \\
W_{m,k\mp2} &= [0 \quad 0 \quad 1]\{\delta_{m,k\pm2}\}
\end{aligned}\right\} \qquad (12)$$

4 - Free edge $\{N_\phi = 0 \;, N_{\phi x} = 0 \;, M_\phi = 0 \;, \bar{Q}_\phi = 0\}$

$$\bar{Q}_\phi = Q_\phi + \frac{1}{L} M^{\cdot}_{\phi x} = Q_\phi + \frac{\lambda}{R} M^{\cdot}_{\phi x}$$

$$\left.\begin{aligned}
U_{m,k\mp 1} &= [\,0 \quad \pm\psi_1^9 \quad \psi_2^9\,]\{\delta_{m,k}\} + [\,1 \quad 0 \quad -\psi_3^9\,]\{\delta_{m,k\pm 1}\} \\
V_{m,k\mp 1} &= [\,\mp\psi_4^9 \quad 0 \quad \pm\psi_5^9\,]\{\delta_{m,k}\} + [\,0 \quad 1 \quad 0\,]\{\delta_{m,k\pm 1}\} \\
W_{m,k\mp 1} &= [\,0 \quad 0 \quad \psi_6^9\,]\{\delta_{m,k}\} + [\,0 \quad 0 \quad -1\,]\{\delta_{m,k\pm 1}\} \\
W_{m,k\mp 2} &= [\,0 \quad \mp\psi_7^9 \quad \psi_8^9\,]\{\delta_{m,k}\} + [\,0 \quad 0 \quad \psi_9^9\,]\{\delta_{m,k\pm 1}\} \\
&\quad + [\,0 \quad 0 \quad 1\,]\{\delta_{m,k\pm 2}\}
\end{aligned}\right\} \quad (13)$$

where

$$C_1^{10} = \frac{1-\nu}{2}\eta\alpha \;, C_3^{10} = 2\alpha^3 - \{1-(2-\nu)\eta^2\}\alpha \;, \quad C_5^{10} = 3(1-\nu)\eta^2 \;,$$

$$\psi_1^9 = C_4^1/C_1^4 \;, \psi_2^9 = C_3^9 C_6^6 / C_3^6 C_1^4 \;, \quad \psi_3^9 = 2C_3^9/C_1^4 \;,$$

$$\psi_4^9 = C_4^2/C_2^2 \;, \psi_5^9 = (C_6^2 - C_3^2\psi_6^9)/C_2^2 \;, \quad \psi_6^9 = C_6^6/C_3^6 \;,$$

$$\psi_7^9 = (C_5^{10} + C_1^{10}\psi_1^9)\Delta^9 \;, \psi_8^9 = (C_3^{10}\psi_6^9 - C_1^{10}\psi_2^9)\Delta^9 \quad \text{and} \quad \psi_9^9 = (C_1^{10}\psi_3^9 - 2C_3^{10})\Delta^9$$

NUMERICAL EXAMPLES:

For the purpose of verifying the accuracy and the validity of the Author's nodal line finite difference method NLFDM in the analysis of circular cylindrical shells, Gibson's simplified analytic solution is considered as a reference. Gibson's solution is based on the Schorer's characteristic equation taking into consideration the following simplifications

$$\nu = 0 \quad , \quad M_x = 0 \;, \quad M_{x\phi} = M_{\phi x} = 0 \;, \quad Q_x = 0$$

$$\varepsilon_\phi = \frac{1}{R}(v^{\cdot} - w) = 0 \;, \quad \varepsilon_{x\phi} = \frac{1}{R}(u^{\cdot} + \lambda v^{\cdot}) = 0$$

It is worth noting that Gibson's solution has acceptable accuracy for shallow circular cylindrical shells only.

Herein, two numerical examples are chosen to demonstrate the accuracy and the validity of the nodal line finite difference method in the analysis of circular cylindrical shells in its general form.

Example 1 : A simply supported thin elastic isotropic circular cylindrical shell with the dimensions shown in Fig. 3 has been analyzed by using the proposed technique. The dimensions of shell are chosen to go on with the definition of shallow shells (Equation (4.1), page 146, Ref. [12]). The shell is assumed to be subjected to a distributed load with the intensity of its own weight. The two longitudinal edges of the shell are assumed to be free. Due to symmetry in geometry and loading condition in the circumferential direction, only half of the shell divided into thirty-one nodal lines ($\Delta\phi = 40^\circ/30 = 1^\circ 20'$) is considered in the analysis. Poisson's ratio ν is assumed to be equal zero to match Gibson's assumption. The analysis was carried out for one term as well as for five terms of the used basic functions. The obtained results are presented in Tables 1 and 2 together with the results obtained from Gibson's solution.

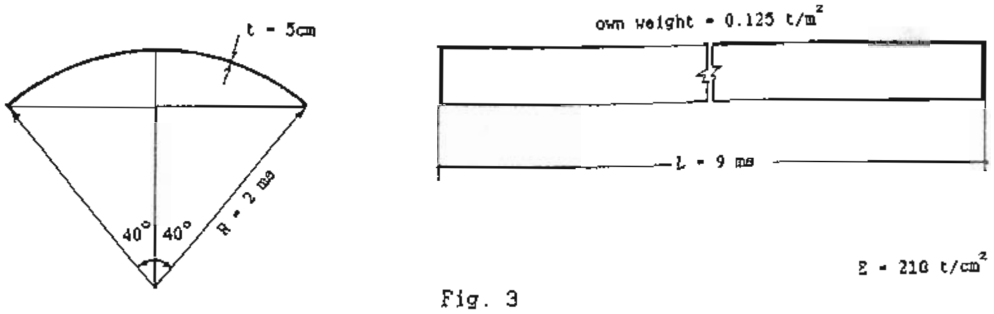

Fig. 3

Table 1. :Displacements and internal forces at x = L/2

		φ											No of terms	SOURCE
		0	4	8	12	16	20	24	28	32	36	40		
v	mm	0.0000	0.3859	0.7723	1.1595	1.5475	1.9361	2.3250	2.7130	3.0990	3.4813	3.8583	5	NLFDM
		0.0000	0.3861	0.7729	1.1609	1.5507	1.9422	2.3348	2.7278	3.1199	3.5096	3.8949	1	NLFDM
		0.0000	0.1734	0.3577	0.5635	0.8007	1.0783	1.4046	1.7855	2.2260	2.7294	3.2969	1	Gibson [12]
w	mm	5.5336	5.5387	5.5471	5.5585	5.5692	5.5746	5.5695	5.5486	5.5073	5.4421	5.3512	5	NLFDM
		5.5387	5.5438	5.5582	5.5791	5.6023	5.6224	5.6333	5.6288	5.6038	5.5542	5.4779	1	NLFDM
		4.9172	4.9394	5.0049	5.1109	5.2523	5.4231	5.6165	5.8255	6.0439	6.2667	6.4908	1	Gibson [12]
N_x	t/m	-10.7325	-10.4119	-9.4499	-7.8454	-5.5971	-2.7030	0.8395	5.0334	9.8823	15.3909	21.5667	5	NLFDM
		-10.9493	-10.6305	-9.6717	-8.0667	-5.8050	-2.8729	0.7461	5.0709	10.1215	15.9189	22.4847	1	NLFDM
		-10.6374	-10.3305	-9.4072	-7.8593	-5.6736	-2.8327	0.6849	4.9030	9.8478	15.5463	22.0264	1	Gibson [12]
N_ϕ	t/m	-0.4585	-0.4492	-0.4216	-0.3767	-0.3187	-0.2508	-0.1817	-0.1121	-0.0540	-0.0120	0.0000	5	NLFDM
		-0.5558	-0.5445	-0.5112	-0.4570	-0.3865	-0.3039	-0.2189	-0.1340	-0.0632	-0.0134	0.0000	1	NLFDM
		-0.5651	-0.5533	-0.5186	-0.4632	-0.3909	-0.3066	-0.4172	-0.1311	-0.0580	-0.0098	0.0000	1	Gibson [12]
M_x	tm/m	0.0158	0.0158	0.0156	0.0153	0.0149	0.0144	0.0138	0.0131	0.0123	0.0113	0.0102	5	NLFDM
		0.0159	0.0159	0.0158	0.0157	0.0155	0.0153	0.0149	0.0145	0.0139	0.0131	0.0123	1	NLFDM
		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1	Gibson [12]
M_ϕ	tm/m	-0.0373	-0.0364	-0.0338	-0.0297	-0.0245	-0.0187	-0.0127	-0.0074	-0.0032	-0.0006	0.0000	5	NLFDM
		-0.0419	-0.0409	-0.0379	-0.0332	-0.0272	-0.0204	-0.0136	-0.0075	-0.0029	-0.0003	0.0000	1	NLFDM
		-0.0501	-0.0489	-0.0454	-0.0400	-0.0330	-0.0253	-0.0175	-0.0104	-0.0048	-0.0012	0.0000	1	Gibson [12]
Q_ϕ	t/m	0.0000	0.0128	0.0247	0.0341	0.0410	0.0441	0.0419	0.0360	0.0257	0.0130	0.0015	5	NLFDM
		0.0000	0.0153	0.0295	0.0408	0.0489	0.0524	0.0497	0.0423	0.0297	0.0140	0.0036	1	NLFDM
		0.0000	0.0170	0.0324	0.0449	0.0533	0.0565	0.0542	0.0464	0.0336	0.0174	0.0000	1	Gibson [12]

Table 2. :Displacements and internal forces at x = 0

		φ											No of terms	SOURCE
		0	4	8	12	16	20	24	28	32	36	40		
u	mm	0.3006	0.2920	0.2660	0.2225	0.1610	0.0809	-0.0186	-0.1381	-0.2788	-0.4416	-0.6279	5	NLFDM
		0.2985	0.2898	0.2637	0.2199	0.1582	0.0782	-0.0206	-0.1386	-0.2764	-0.4345	-0.6137	1	NLFDM
		0.2902	0.2819	0.2567	0.2144	0.1548	0.0773	-0.0187	-0.1330	-0.2687	-0.4242	-0.6010	1	Gibson [12]
$N_{x\phi}$	t/m	0.0000	0.5613	1.0969	1.5796	1.9792	2.2607	2.3820	2.2916	1.9246	1.1987	0.0073	5	NLFDM
		0.0000	0.5287	1.0265	1.4617	1.8028	2.0175	2.0727	1.9344	1.5678	0.9369	0.0050	1	NLFDM
		0.0000	0.5135	0.9970	1.4203	1.7527	1.9627	2.0179	1.8846	1.5281	0.9124	0.0000	1	Gibson [12]
$N_{\phi x}$	t/m	0.0000	0.5397	1.0939	1.5752	1.9737	2.2543	2.3750	2.2843	1.9172	1.1913	0.0000	5	NLFDM
		0.0000	0.5278	1.0245	1.4589	1.7993	2.0134	2.0682	1.9297	1.5629	0.9320	0.0000	1	NLFDM
		0.0000	0.5135	0.9970	1.4203	1.7527	1.9627	2.0179	1.8846	1.5281	0.9124	0.0000	1	Gibson [12]
$M_{x\phi}$	tm/m	0.0000	-0.0031	-0.0062	-0.0089	-0.0112	-0.0130	-0.0142	-0.0149	-0.0150	-0.0148	-0.0147	5	NLFDM
		0.0000	-0.0020	-0.0039	-0.0057	-0.0072	-0.0083	-0.0091	-0.0096	-0.0099	-0.0100	-0.0100	1	NLFDM
		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1	Gibson [12]
$M_{\phi x}$	tm/m	0.0000	-0.0031	-0.0060	-0.0087	-0.0110	-0.0128	-0.0140	-0.0146	-0.0148	-0.0147	-0.0147	5	NLFDM
		0.0000	-0.0020	-0.0038	-0.0055	-0.0070	-0.0081	-0.0089	-0.0094	-0.0097	-0.0099	-0.0100	1	NLFDM
		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1	Gibson [12]
Q_x	t/m	-0.0165	-0.0158	-0.0139	-0.0106	-0.0063	-0.0012	0.0043	0.0096	0.0137	0.0156	0.0145	5	NLFDM
		-0.0087	-0.0083	-0.0073	-0.0058	-0.0038	-0.0017	0.0004	0.0022	0.0035	0.0039	0.0035	1	NLFDM
		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1	Gibson [12]

Example 2 : A simply supported free edged circular cylindrical shell having the dimensions shown in Fig. 4 has been analyzed. The shell is assumed to be subjected to concentrated line load at the crown. Due to symmetry in the circumferential direction, only half of the shell, divided into thirty-one nodal lines ($\Delta\phi = 60^\circ/30 = 2^\circ$), is considered in the analysis. Two values of Poisson's ratio, ν, are considered; these are 0 and 1/6. The analysis was carried out for five terms of the used basic functions. The obtained results are presented in Tables 3 and 4. Compared with the definition of shallow shells, this shell is considered deep. Only the results of the nodal line finite difference technique are presented, since Gibson's solution is not acceptable for deep shells and not valid for concentrated line loading as well.

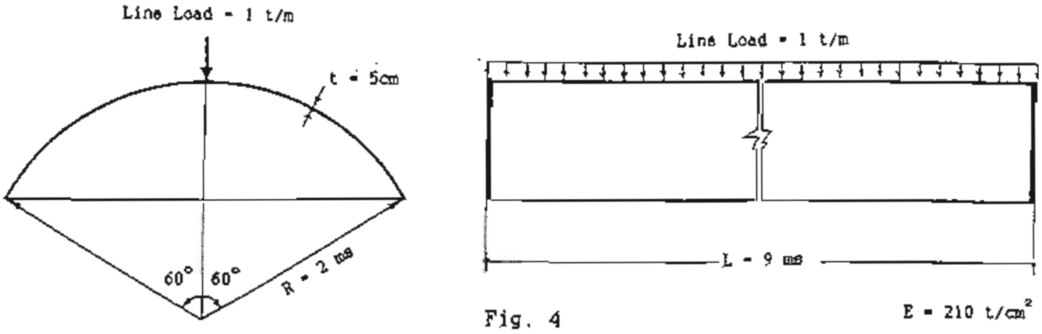

Fig. 4

E = 210 t/cm²

Table 3. :Displacements and internal forces at x = L/2

		φ											Poisson's ratio	SOURCE
		0	6	12	18	24	30	36	42	48	54	60		
v	mm	0.0000 0.0000	0.3595 0.3565	0.6712 0.6655	0.9031 0.8947	1.0364 1.0248	1.0613 1.0455	0.9748 0.9532	0.7781 0.7484	0.4749 0.4344	0.0704 0.0161	-0.4299 -0.5011	0 1/6	XLFDM
w	mm	3.5385 3.4865	3.2842 3.2350	2.6491 2.6049	1.7773 1.7366	0.7742 0.7330	-0.2852 -0.3322	-1.3505 -1.4090	-2.3898 -2.4654	-3.3852 -3.4830	-4.3279 -4.4522	-5.2147 -5.3684	0 1/6	
N_x	t/m	-7.4045 -7.3336	-7.0325 -6.9203	-5.9734 -5.8415	-4.3444 -4.2115	-2.2931 -2.1772	0.0010 0.0811	2.3299 2.3538	4.4669 4.4127	6.1805 6.0736	7.2459 6.9593	7.4536 7.0083	0 1/6	
N_ϕ	t/m	-1.3964 -1.3899	-1.1427 -1.1363	-0.9166 -0.9114	-0.7202 -0.7164	-0.5548 -0.5518	-0.4143 -0.4121	-0.2926 -0.2919	-0.1853 -0.1859	-0.0944 -0.0959	-0.0274 -0.0290	0.0000 0.0000	0 1/6	
M_x	tm/m	0.0094 0.0543	0.0084 0.0375	0.0067 0.0244	0.0048 0.0145	0.0027 0.0071	0.0004 0.0015	-0.0020 -0.0028	-0.0046 -0.0060	-0.0071 -0.0084	-0.0096 -0.0104	-0.0119 -0.0121	0 1/6	
M_ϕ	tm/m	0.2649 0.2704	0.1701 0.1755	0.1022 0.1072	0.0546 0.0592	0.0230 0.0270	0.0038 0.0072	-0.0061 -0.0033	-0.0092 -0.0071	-0.0079 -0.0066	-0.0044 -0.0037	0.0000 0.0000	0 1/6	
Q_ϕ	t/m	0.0000 0.0000	0.3893 0.3897	0.2795 0.2803	0.1938 0.1952	0.1273 0.1289	0.0763 0.0783	0.0387 0.0408	0.0131 0.0151	-0.0023 -0.0002	-0.0090 -0.0072	-0.0107 -0.0093	0 1/6	

Table 4. :Displacements and internal forces at x = 0

		φ											Poisson's ratio	SOURCE
		0	6	12	18	24	30	36	42	48	54	60		
u	mm	0.2448 0.2325	0.2297 0.2179	0.1878 0.1773	0.1264 0.1176	0.0538 0.0471	-0.0217 -0.0260	-0.0921 -0.0939	-0.1498 -0.1490	-0.1673 -0.1833	-0.1970 -0.1893	-0.1712 -0.1588	0 1/6	NLFDM
$N_{x\phi}$	t/m	0.0000 0.0000	1.2865 1.2715	2.2547 2.2238	2.7857 2.7438	2.8953 2.8475	2.6640 2.6141	2.1897 2.1402	1.5708 1.5244	0.9108 0.8716	0.3306 0.3057	-0.0163 -0.0148	0 1/6	
$N_{\phi x}$	t/m	0.0000 0.0000	1.3043 1.2865	2.2778 2.2434	2.8087 2.7634	2.9164 2.8657	2.6832 2.6308	2.2074 2.1558	1.5877 1.5394	0.9273 0.8864	0.3470 0.3205	0.0000 0.0000	0 1/6	
$M_{x\phi}$	tm/m	0.0000 0.0000	0.0354 0.0298	0.0461 0.0389	0.0458 0.0389	0.0419 0.0360	0.0380 0.0330	0.0353 0.0309	0.0337 0.0298	0.0330 0.0294	0.0327 0.0295	0.0327 0.0297	0 1/6	
$M_{\phi x}$	tm/m	0.0000 0.0000	0.0356 0.0300	0.0463 0.0392	0.0461 0.0393	0.0422 0.0363	0.0383 0.0333	0.0355 0.0311	0.0339 0.0300	0.0331 0.0296	0.0328 0.0295	0.0327 0.0297	0 1/6	
Q_x	t/m	0.2968 0.2988	0.1199 0.1218	0.0261 0.0279	-0.0154 -0.0139	-0.0274 -0.0261	-0.0254 -0.0243	-0.0191 -0.0181	-0.0135 -0.0125	-0.0102 -0.0093	-0.0088 -0.0079	-0.0079 -0.0070	0 1/6	

CONCLUSION

The present work contributes to the establishment and the extension of the use of the Author's nodal line finite difference method to include the analysis of thin elastic isotropic circular cylindrical shells. In the present analysis, the shell is assumed to be simply supported at the two opposite end diaphragms. The proposed method is considered a semi-analytical procedure which transforms the three governing partial differential equations into three simultaneous nodal line difference equations by means of using analytical technique in the longitudinal direction while in the circumferential direction, the differential operators are replaced by the well known difference expressions. The method has the advantage of being simple and straight forward in formulation, so that, it treats the governing partial differential equations in its complex mathematical form without any simplifications or assumptions that affect the generality of these equations. Therefore, the present formulation covers the analysis of short, intermediate, long, closed, open shallow and open deep circular cylindrical shells. Moreover, the elements of the final square matrix are symmetrically distributed around the diagonal in a banded form with small half band width and this reduces to a great extent the storage requirements and the computation time. The method can be easily extended to include the dynamic analysis as well as the stability of circular cylindrical shells.

APPENDIX I

NOTATION

u , v , w - displacement components.

$U_{m,k}$, $V_{m,k}$, $W_{m,k}$ - nodal line parameters.

L - length of the shell.

$2\phi_o$ - total angle of the shell.

$\Delta = \Delta\phi$ - constant angle between the nodal lines.

E - modulus of elasticity.

t - thickness of the shell.

ν - Poisson's ratio.

D - in-plane stiffness of the shell.

B - bending stiffness of the shell.

X , Y , Z - load intensity components.

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