GENERALIZATION OF FINITE INTEGRAL TRANSFORMS FOR TREATING NONLINEAR PROBLEMS IN HEAT DIFFUSIÓN. PART I: METHODOLOGY "Linear Case"

تعميم التحويل التكاملي المحدود لحل
معادلات الانتقال الحراري الغير خطية
الجزء الأول ـ طريقة الحل

Bishri Abdel-Hamid
Assistant Professor
Applied Mathematics and Physical Sciences Dept.,
Al-Mansoura University, Egypt

م هذا البحث طريقة موحده لحل المعادلات التفاضلية الجزئية
غير خطية والتي تظهر بشكل عام في مجال انتقال الحرارة والمادة و
اميكا الموائع وغيرها. والطريقة مبنية على طرق التحويل
تكاملي المستخدمة لحل المعادلات التفاضلية الخطية مع تطوير في
تخدام لمعالجة الحالات الغير خطية .

ي هذا البحث عرض لطريقة التحويل التكاملي المقترحة وبغرض اختبار
رق بينها وبين طرق التحويل التكاملي التقليدية تم التطبيق على
ادلة تفاضلية خطية خاضعة لشروط حدية قابلة للفصل لكن بصعوبة
موسعة . ولقد اختيرت هذه المسألة البسيطة لبيان صعوبة استخدام
طرق التقليدية . وفي نهاية البحث عرض لنتائج طريقتي الحل مع مقارنة
حلول في حالات مختلفة واختبار جدوى الطريقة المقترحة .

ABSTRACT

A general analytic methodology, based on the finite integr transform, which is capable of resolving diffusion proble described by nonlinear partial differential equations accurate and rapidly has been introduced previously by several author The method is extended in this work to solve partial differenti equations with nonseparable and/or nonlinear boundary condition The analysis and merit of the proposed method is demonstrated a heat diffusion problem of practical interest and the result compared to the exact solution of the conventional fini integral transform method.

INTRODUCTION

In the field of linear heat transfer, the transient he diffusion equation is linearized by considering the therm properties to be independent of temperature; furthermore, t boundary conditions are also taken to be linear. This class linear, transient heat diffusion has been treated in detai

[1,2]. However, when the thermophysical properties [3-6] and/or the volumetric heat source [1,7] become temperature dependent, the field equation becomes nonlinear and the solutions can not be obtained by any of the elegant methods presented in [1,2,8]. In addition, if the temperature level becomes high, radiation and/or change of phase may occur, and, as a result, the boundary conditions become nonlinear [9,10]. The transition from a linear to a nonlinear model introduces additional mathematical and computational difficulties which have to be dealt with. As it appears, solving nonlinear problems analytically is rare and each nonlinear problem requires a special treatment.

The finite integral transform method is one standard method used in solving heat and mass diffusion problems. Ozisik [1], Luikov [11] and Mikhailov and Ozisik [12] have demonstrated the usefulness and effectiveness of the method in solving linear, separable partial differential equations. Frankel and Vick [13] have generslized the finite integral transform technique to the solution of nonlinear diffusion equations having a temperature dependent thermal conductivity. Frankel [14] has demonstrated the accuracy of the method.

The current work introduces an analytic methodology based on the finite integral transform for soiving partial differential equations with nonseparable and/or nonlinear boundary conditions. The work is presented in two parts. This part, part I, focuaes on the development and merit of the methodology by solving a heat diffusion problem the choice of which is two-fold. First, the problem is one of practical importance from the engineering point of view. Secondly, the problem is mathematically described by a differential equation subject to nonhomogeneous boundary conditions for which separation of variables, though troublesome, can be achieved. These mathematical characteristics help in demonstrating the effectlvenese of the method as compared to the conventional use of finite integral transform technique which requires separation of variables. In a subsequent paper, part II, the method will be demonstrated on nonlinear nonseparable problems as in heat diffusion with radiation interaction at the boundaries.

PROBLEM DESCRIPTION

To introduce the method without undue complications, a one-dimensionai transient heat diffusion in a slab of finite thickness having a thin film on one face (at $\eta=0$) and insulated back surface is considered. This problem is discussed by the author in [15] and is mathematically described by a field equation and a set of boundary conditions which can be written in dimensionless form as follows

$$\frac{\partial^2\theta}{\partial\eta^2}(\eta,\tau) = \frac{\partial\theta}{\partial\tau}(\eta,\tau) \ , \qquad (1)$$

subject to

$$\frac{\partial\theta}{\partial\eta}(\eta,\tau) - \beta\theta(\eta,\tau) = \Gamma\frac{\partial\theta}{\partial\tau}(\eta,\tau) \ ; \qquad \eta=0, \ \tau>0 \qquad (2a)$$

$$\frac{\partial\theta}{\partial\eta}(\eta,\tau) = 0 \ ; \qquad \eta=1, \ \tau>0 \qquad (2b)$$

and

$$\theta(\eta,\tau) = F(\eta); \qquad \tau=0, \quad 0<\eta<1. \qquad (2c)$$

Where η, τ, and θ are the dimensionless spatial, time, and field variables, respectively; β is Biot number, and Γ is a heat capacity ratio defined by

$$\Gamma = \frac{(\rho cb)_f}{(\rho cL)_S} \qquad (3)$$

with f and S refer to the film and the solid, respectively.

Solution to the above differential equation subject to the given boundary and initial conditions is developed in the following sections using the conventional and the generalized finite integral transform techniques.

FINITE INTEGRAL TRANSFORM: CONVENTIONAL (EXACT) SOLUTION

In the finite integral transform technique, the integral transform pair needed for the solution of a given problem is developed by considering representation of an arbitrary function in terms of the eigenfunctions corresponding to the given eigenvalue problem. Obtaining the required eigenvalue problem may be accomplished by considering the homogeneous part of the nonhomogeneous field equation and then employing separation of variables to obtain the basis functions.

Employing separation of variablee on the boundary value problem given by equations (1) and (3) yields the following eigenvalue problem

$$\psi''(\eta) + \lambda_n^2 \, \psi(\eta) = 0, \qquad n=1,2,... \qquad (4)$$

subject to

$$\psi'(\eta) = (\beta - \Gamma\lambda_n^2) \, \psi(\eta); \qquad \text{at } \eta=0 \qquad (5a)$$

$$\psi'(\eta) = 0; \qquad \text{at } \eta=1. \qquad (5b)$$

The eigenvalues of this problem is determined from the following transcendental equation

$$\lambda_n \sin(\lambda_n) - (\beta - \Gamma\lambda_n^2) \cos(\lambda_n) = 0. \qquad (6)$$

The eigenfunction corresponding to the n^{th} eigenvalue λ_n is given by

$$\psi_n(\eta) = \lambda_n \cos(\lambda_n \eta) + (\beta - \Gamma\lambda_n^2) \sin(\lambda_n \eta). \qquad (7)$$

The orthogonality property with respect to a weight function $W(\eta)$ may be established as

$$\int_0^1 W(\eta)\psi_n(\eta)\psi_m(\eta)d\eta = \begin{cases} 0, & m \neq n \\ N(\lambda_n), & m = n \end{cases} \qquad (8)$$

where $N(\lambda_n)$ is the normalization integral.The weight function which satisfies the orthogonality relation is derived in the appendix and is given by,

$$W(\eta) = 1 + \Gamma\delta(\eta), \qquad (9)$$

where $\delta(\eta)$ is the *Dirac delta-function*.

The appropriate transform pair can now be defined as [1],

INTEGRAL TRANSFORM

$$\Phi(\lambda_n,\tau) = \int_0^1 W(\eta)\psi(\eta)\theta(\eta,\tau)\, d\eta \qquad (10a)$$

INVERSION FORMULA

$$\theta(\eta,\tau) = \sum_{n=1}^{\infty} \frac{\psi_n(\eta).\Phi(\lambda_n,\tau)}{N(\lambda_n)} \qquad (10b)$$

Following the standard transformation procedures [1], we reduce equation (1), and the boundary and initial conditions given by equations (2a-c) to a system of first order ordinary differential equations in the transform dimensionless temperature, namely

$$\frac{d\Phi}{d\tau}(\lambda_n,\tau) + \lambda_n^2\Phi(\lambda_n,\tau) = 0, \quad n=1,2,... \qquad (11)$$

subject to the following transformed initial condition

$$\Phi(\lambda_n, 0) = \tilde{F}(\lambda_n), \qquad n=1,2,\ldots \tag{12}$$

where $\tilde{F}$ is the transformed initial distribution function.

The general solution to the above system can be written as

$$\Phi(\lambda_n, \tau) = \tilde{F}(\lambda_n) . e^{-\lambda_n^2 \tau} \qquad n=1,2,\ldots \tag{13}$$

Equation (13) is an exact solution of the transformed dimensionlees temperature. Finally, substituting this solution for the transform into the inversion formula yields the solution for the dimenaionless temperature.

FINITE INTEGRAL TRANSFORM: GENERALIZED METHOD

The solution given above was lengthy and difficult because of the necessity of finding the weight function that satisfies the orthogonality relation. This approach declines to lead to an exact solution in cases where the boundary conditions are nonseparable or when the separation process can not be easily performed. This is clearly seen by the case when radiation occurs at a boundary in accordance to the fourth power law.

This paper suggests a method of solution that takes full advantage of the finite integral transform technique by overcoming the obstacle of obtaining a nonstandard weight function.The time dependent term in the first boundary condition, equation (2a), causes most of the difficulties that arise in tbe conventional approach. Treating this term as a "source" and keeping other conditions unchanged, then employing the method of separation of variables on the obtained associated homogeneous prohlem we obtain the following eigenvalue problem

$$\psi''(\eta) + \lambda_n^2 \psi(\eta) = 0, \tag{14}$$

subject to

$$\psi'(\eta) - \beta \psi(\eta) = 0; \qquad \eta=0, \tag{15a}$$

$$\psi'(\eta) = 0 ; \qquad \eta = 1. \tag{15b}$$

for which the eigenvalues are defined by the following transcendental equation

$$\beta \cos\lambda_n - \lambda_n \sin\lambda_n = 0. \qquad n=1,2,\ldots \tag{16}$$

The eigenfunctions can be written as

$$\psi_n(\eta) = \lambda_n \cos\lambda_n\eta + \beta \sin\lambda_n\eta, \quad n=1,2,\ldots \tag{17}$$

In this case the orthogonality relation given by equation (8) is satisfied for the above eigenfunction with a weight function equal unity. With this standard value of the weight function, the transformation pair required for the solution takes the following form;

INTEGRAL TRANSFORM

$$\Phi(\lambda_n,\tau) = \int_0^1 \psi_n(\eta)\theta(\eta,\tau)\, d\eta. \tag{18a}$$

INVERSION FORMULA

$$\theta(\eta,\tau) = \sum_{n=1}^{\infty} \frac{\psi_n(\eta).\Phi(\lambda_n,\tau)}{N(\lambda_n)} \tag{18b}$$

Thus the governing dimensionless equation and the associated conditions can be transformed to the following first order ordinary differential equation

$$\frac{d\Phi}{d\tau}(\lambda_n,\tau) + \lambda_n^2\Phi(\lambda_n,\tau) = -\Gamma\psi_n(0)\frac{\partial\theta}{\partial\tau}(0,\tau), \quad n=1,2,... \tag{19}$$

subject to the following transformed initial condition

$$\Phi(\lambda_n,0) = \int_0^1 \psi_n(\eta)\theta(\eta,0)\, d\eta, \quad n=1,2,... \tag{20}$$

Both the field variable and its transform appear in the system given above. This difficulty can be reconciled by utilizing the inversion formula into (19).

Truncation of the set of ordinary differential equations after some finite number of equations is performed and the solution is obtained numerically. However, the numerical treatment requires special care as the resulting system is stiff. Moreover, increasing the number of eigenvalues, which is desired to achieve better accuracy, increases the stiffness of the system. Once the transformed dimensionless temperature Φ is determined, the dimensionless temperature, θ can be reconstructed through the inversion formula.

RESULTS

A qualitative comparison of the two solutions is performed at first for the limiting case where no film is present. In such case, ($b_f=\Gamma=0$), the right-hand side of equation (19) equals zero and the system of first order differential equations of both solutions are identical. Moreover, the transcendental and the eigenfunctions expression of the two solutions become equivalent, i.e, the proposed method gives the exact solution at the limiting case when the film thickness is set equal to zero.

Secondly, the two solutions are compared for values of the heat capacity ratio, Γ equals 0, 0.01, 0.1 and 1 and values of Biot number, β equals 0.05, 0.1, and 1. Differences in dimensionless temperature are evaluated at the front surface (η=0), at the middle (η=0.5), and at the back surface (η=1). Solutions of both methods were obtained using forty eigenvalues in all cases. The solution of the generalized finite integral transform method was computed by implementing an implicit Runge-Kutta method of the third order.

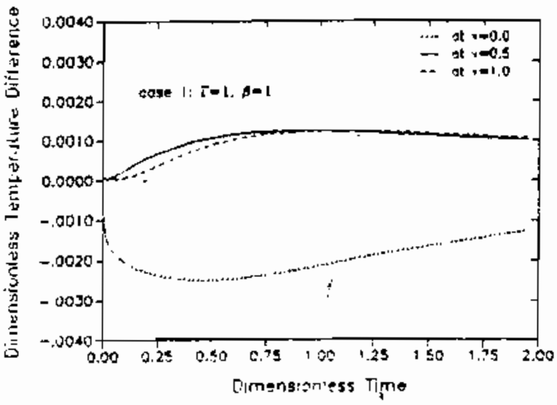
The dimensionless temperature difference and the relative error plots are provided for samples of the results in Figura (1). The minor errors that appear are due to truncation and round off errors. Same results are displayed in tabular form at selected values of time.

In all the cases considered, the difference in dimensionless temperature between the two solutions was found to be less than 0.0003 at both the middle section and the back surface.

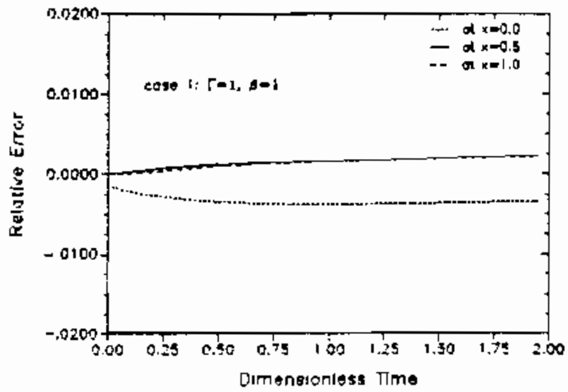
At the front surface, however, the difference between the two methods is related to the Biot number. For values of Biot number equal to 0.05 and 0.1, the dlfference in dimensionless temperature was found to be less than 0.00092 while the relative error remained within the 1% margin. When the Biot number is equal 1.0, the difference between the two solutions at the front surface increased, however, the maximum value of the dimensionless temperature difference remains within 0.007 with a relatlve error within 1%. It is also noticed that the relatlve error is nearly constant over the time domain at each section and the difference in the dimensionless temperature is maximum for early time (small Fourier number) and decays as the time increases.

CONCLUSION

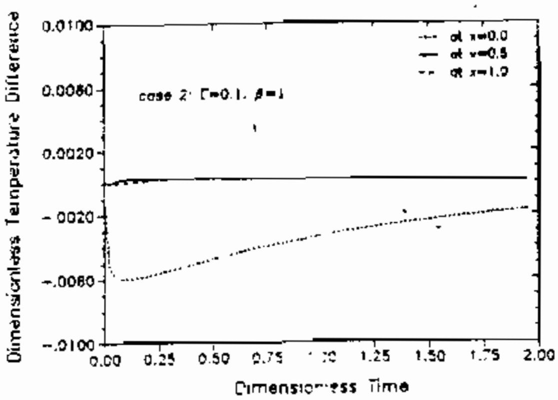
The presented straightforward use of the finite integral transform reduces the mathematical difficulties which arise when difficult nonhomogeneous, separats boundary conditions are encountered. The method yielded similar results as the true exact analytic solution for all considered cases. The application of this method to nonlinear problems will be demonstrated in part II of this study.



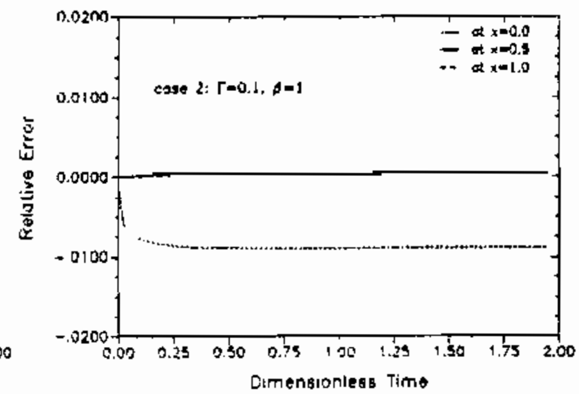
Dimensionless Temperature Difference between the Exact and Generalized Finite Integral Transform Solution Methods



Relative Error of Dimensionless Temperature Obtained from the Exact and Generalized Finite Integral Transfrom Solution Methods



Dimensionless Temperature Difference between the Exact and Generalized Finite Integral Transform Solution Methods



Relative Error of Dimensionless Temperature Obtained from the Exact and Generalized Finite Integral Transfrom Solution Methods

Fig.(1) Dimensionless Temperature Difference and Relative Error Obtained from Exact and Generalized Solution Methods.

NOMENCLATURE

b	Film thickness
c	Specific heat
f	A subscript denotes fluid
h	Heat transfer coefficient between the film and the surrounding
k	Thermal conductivity of the slab
L	Thickness of a slab
$N(\lambda_n)$	Normalization integral
t	Dimensional time variable
T_∞	Ambient temperature
$T(x,t)$	Dimensional Temperature
$W(\eta)$	A weight function
x	Space variable
s	A subscript refers to solids

GREAK SYMBOLS

α	Thermal diffusivity
$\beta=\frac{hL}{k}$	Biot number
$\Phi(\lambda_n,\tau)$	Transform of dimension less temperature
$\eta=\frac{x}{L}$	Dimensionless space variable
λ_n	Eigenvalues
θ_∞	Dimensionless temperature of the surrounding
$\theta(\eta,\tau)=\frac{T(x,t)-T_\infty}{T_o-T_\infty}$	Dimensionless temperature
ρ	Density of solid
$\tau=\frac{\alpha t}{L^2}$	Dimenaionless time
$\psi_n(\eta)$	Eigenfunctions
Γ	Heat capacity ratio

DIMENSIONLESS PARAMETERS

$\eta=\frac{x}{L}$

$\tau=\frac{\alpha t}{L^2}$

$\theta=\frac{T(x,t)-T_\infty}{T_o-T_\infty}$

$\beta=\frac{hL}{k}$

REFERENCES

1. Ozisik, M.N., Heat Conduction, *wiley and Sons, New York*, 1980.

2. Carslaw, H.S., and, J.C. Jaeger, Conduction of Heat in Solids, *2nd. Ed., Oxford Univ. Press, London,* 1959.

3. Thomton, E.A. and Wieting, A.R., "A Finite Element Thermal Analysis Procedure for several Temperature-Dependent Parameters", *J. Heat Transfer, Vol 100,* pp. 551-553, Aug 1978.

4. Mehta, R.C. "On the Solution of Transient Conduction with Temperature-Dependent Thermal Conductivity", *J. Heat Transfer, TRANS ASME, Vol.99*, 1977, pp. 137-139.

5. Orivuori, S. "Efficient Method for Solution on Nonlinear Heat Conduction Problems", *Int. J. Num. Meth. in Engng., Vol 14*, 1461-1476, 1979.

6. Imber, M., "Thermally Symmetric Nonlinear Heat Transfer in Solids," *J. Heat Transfer, Vol. 103*, 745-752, 1981.

7. Na, T.Y., and S.C. Tang, "A Method for the Solution of Conduction Heat Transfer with Nonlinear Heat Generation," *ZAMM, Vol. 49*, 45-52, 1969.

8. Arpaci, V.S., Conduction Heat Transfer, *Addison-Wesley Publ. Co.*, Reading, Mass., 1966.

9. Danielyan, Y.S., and B.G. Aksenov, "Approximate Solution of Nonlinear Problems of Radiant Heat Transfer," *Heat Transfer-Soviet Research, Vol. 15, #3*, 44-52, 1983.

10. Koizumi, M., M.Utamura and, K. Kotani, "Three Dimentional Transient Heat Conduction Analysis with Non-linear Boundary Conditions by Boundary Element Methods," *J. Nucl. Sci. Technol, Vol. 22, #12*, PP 972-982, 1985.

11. Liukov, A. V. "Analytical Heat Diffusion Theory," ed. by S.P. Hartnett, *Academic Press (N.Y.)*, 1968.

12. Mikhailov, M. D. and M. N. Ozisik, Unified Analysis and Solutions of Heat and Mass Diffusion, *John Wiley & Sons, Inc. (N.Y)*, 1984.

13. Frankel, J.I., and B. Vick, "An Exact Methodology for Solving Nonlinear Diffusion Equations Based on Integral Transforms," *Appl. Num. Math., Vol 3*, 1987, pp467-477.

14. Frankel, J.I., "The Use of Integral Transforms in Nonlinear Diffusion Equations," *National Heat Transfer Conf., Pittsburgh, PA,* 1987.

15. Abdel-Hamid, B.,"An Integral Transform Methodology for Solving Nonlinear Diffusion Equations," *A Ph.D. Dissertation,* Florida Inst. of Tech., Melbourne, FL, 1989.

APPENDIX

DERIVATION OF THE WEIGHT FUNCTION GIVEN BY EQUATION (9)

We start by writing equation (5) for two different eigen-values, say λ_m and λ_n, then cross multiplying by ψ_m and ψ_n, subtracting the results and rearranging to get

$$\psi_m\psi_n(\lambda_m^2-\psi_n^2) = \left[\psi_n'\psi_m - \psi_n\psi_m' \right]' . \tag{A.1}$$

Integrating over η, $\eta\in[0,\eta]$, and making use of the boundary conditions given by equation (3b), yields

$$(\lambda_m^2-\psi_n^2)\int_0^1 \psi_m\psi_n d\eta = \psi_n(0)\psi_m'(0) - \psi_n'(0)\psi_m(0). \tag{A.2}$$

Utilizing equation (4) along witb equation (3a), equation (A.2) can be further reduced to

$$\int_0^1 \psi_m\psi_n d\eta = -\Gamma\lambda_n\lambda_m . \tag{A.3}$$

Assuming the following expression for the weight function

$$W(\eta) = 1 + \Phi(\eta), \tag{A.4}$$

where $\Phi(\eta)$ is unknown function of (η), the orthogonality relation given by (8) is then satisfied when

$$\int_0^1 [1+\Phi(\eta)]\psi_m\psi_n \, d\eta = 0. \tag{A.5}$$

Noticing from equation (6) that $\lambda_n = \psi_n(0)$ and $\lambda_m = \psi_m(0)$, we introduce (A.3) into (A.5) to get

$$\int_0^1 \Phi(\eta)\psi_m\psi_n d\eta = \Gamma\psi_m(0)\psi_n(0). \tag{A.6}$$

Making use of the sifting properties of the *Dirac delta*-function, we rewrite equation (A.6) as

$$\int_0^1 \Phi(\eta)\psi_m\psi_n d\eta = \int_0^1 \Gamma\delta(\eta)\psi_m(\eta)\psi_n(\eta)d\eta. \tag{A.7}$$

From (A.7) we observe that $\Phi(\eta) = \Gamma\delta(\eta)$ and the desired weight function takes form

$$W(\eta) = 1 + \Gamma\delta(\eta). \tag{A.9}$$

Which is the function used in the analysis