كلفات + الاتاج + حديث

Mansoura University
Faculty of Engineering
Math. and Eng. Phy. Dept.

Mathematics 4

1st year, Electrical Engineering Dept.
Final Exam, May 2014
Time Allowed: 3 Hours

الامتحان في صفحتين. برجاء بدء إجابة كل فرع من إحدى نهايتي ورقة الاجابة

Question 1 [30 point]

a. **[10 pts]** If solution of $(1-x^2)y''(x) - 2xy'(x) + 20y(x) = 0$

where $y(0) = 3$, $y'(0) = 0$ can be expressed by power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find

i) The recurrence relation for a_n ii) a_6 , a_5 , a_4 , a_3 , a_2 , a_1 , a_0

b. **[10 pts]** Prove that $\int_0^\infty \frac{\cos(x)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$ (Hint: use the relation $\int_0^\infty t^{\frac{-1}{2}} e^{-xt} dt = \frac{\Gamma(\frac{1}{2})}{\sqrt{x}}$.

Then multiply both side by $\cos(x)$ and integrate with respect to x from 0 to ∞)

c. **[5 pts]** Find $I = \int_0^{\frac{\pi}{2}} x^2 J_{\frac{1}{2}}^4(x) dx$

d. **[5 pts]** Show that $y(x) = \sqrt{x} J_{\frac{1}{2}}(x)$ is solution of $x^2 y''(x) + (x^2 - 2)y(x) = 0$

Question 2 [25 point]

a. **[10 pts]** Expand the function $f(x) = |x|$ where $-1 \le x \le 1$ in term of Legendre polynomials. Find the first two nonzero terms

b. **[15 pts]** If $f(x) = x - x^2$ where $0 \le x \le 1$. Find

i) Fourier Cosine Series. Then prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

ii) Fourier Sine Series. Then prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$

iii) <u>Without Integration</u> if $f(x) = \begin{cases} x - x^2 & 0 \le x \le 1 \\ x^2 - x & -1 \le x \le 0 \end{cases}$. Find

$$I = \int_{-1}^{1} f(x) \sin(5\pi x) \cos(4\pi x) dx$$

<u>You can use the following relations through the exam:</u>

$P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{1}{2}(3x^2 - 1)$ $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$ $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ $0 < x < 1$

$\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}(t) \cos^{2y-1}(t) dt = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. (a) Show that $u(x, y) = x + e^{y} \cos x$ is harmonic. Find an analytic function $f(z) = u(x, y) + iv(x, y)$ as a function of z. [10 pts]

(b) Show that $|\sinh y| \leq |\sin z| \leq \cosh y$. Describe the points z at which: [10 pts]

i. $|\sin z| = |\sinh y|$.

ii. $|\sin z| = \cosh y$.

(c) i. Let $z = x + iy$. Define each of the following: [10 pts]

$-z$, $\bar{z}$, $|z|$, $\operatorname{Arg} z$ and $\arg z$.

ii. Consider the n^{th} roots w_k of the complex number $z = -1$.

1. Write the algebraic properties of w_k.
2. Write the geometric properties of w_k.
3. There are some properties of w_k which depend on the evenness or oddness of n, write some of these properties.

4. (a) Evaluate the following without using the residue theorem: [10 pts]

i. $\int_C (e^{\cos z} \sin z + \bar{z}) dz$, where c is the line segment from $z = 0$ to $z = \frac{\pi}{2} + i\frac{\pi}{2}$. Put your answer in the form $a + ib$.

ii. $\int_{|z|=2} \left(e^{z^2} + \frac{\cos z}{z(z-1)^3}\right) dz$.

(b) Use the residue theorem to evaluate [10 pts]

$$\int_{|z|=2} \left(\frac{\sin z}{z\left(z-\frac{\pi}{2}\right)^2} + \frac{3}{2}(z-1)^3 e^{\frac{1}{z-1}} \right) dz.$$

(c) Consider the function f given by $f(z) = \frac{1}{(z+a)(z-b)}$, $|a| < |b|$. [10 pts]

i. Expand f in a Laurent series valid in $|a| < |z| < |b|$.

ii. Use the Laurent series expansion obtained above to calculate

$\int_{|z|=\frac{|a|+|b|}{2}} z^{10} f(z) dz$ and $\frac{d^{10} f(0)}{dz^{10}}$.

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Useful relations:

1) $\int x^n J_{n-1}(x)dx = x^n J_n(x) + c.$

2) $\int x^{-n} J_{n+1}(x)dx = -x^{-n} J_n(x) + c.$

3) $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$

4) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x).$

5) $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x).$