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PROJECTION GEOMETRY OVER RINGS

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ABSTRACT

A generalization of the fundamental theorem of projective geometry is established for non-injective mappings between projective geometries defined over rings which satisfy a stable range condition. A geometrical characterization of the stable range condition is also given.

1. INTRODUCTION

Let A and B be associative rings with identity and M and N free right modules of finite rank at least three over A and B, respectively. Assume both A and B have the invariant basis property so that the rank of M, or of N, is and invariant of the module (see Cohn[2]; in particular, if A is commutative or right noetherian it has the invariant basis property).

Assume $n=\text{rank } M= \text{rank } N\geq 3$.

Let P(M) be the projective space of all point and lines of M, that is , the set of all free rank one direct summands of M, called the *points* of P(M), together with all free rank two direct summands of M, the *lines* of P(M).

Let P(N) be the corresponding projective space of N. Two points P_1 and P_2 of P(M) generate the line L of P(M) if $P_1 + P_2 = L$.

Definition A mapping $\pi : P(M) \rightarrow P(N)$ is projectivity if it carries points and lines of P(M) into points and lines of P(N), respectively, and, moreover, if P_1 and P_2 are points generating the line L of P(M), then πP_1 and πP_2 generate πL.

A set of points $P_1,\ldots, P_m$ if P(M) span (M) if $P_1 + \ldots + P_m = M$, that is, M is the module sum of the submodules $P_1,\ldots , P_m$. These points form a projective frame of P(M) if the number m is minimal (hence m = n = rank M). Then each $P_i + P_j$, $i \neq j$, is a line of P(M). The projective frame is thus a set of n points in general position.

FUNDAMENTAL THEOREM

Assume the stable range condition SRn(A) holds, π: P(M) $\rightarrow$ P(N) is a projectivity and there exists a projective frame $\Im$ of P(M) whose image $\pi\Im$ under π is a projective frame of P(N). Then there exists a ring homomorphism φ: A$\rightarrow$B and a φ-semilinear mapping β: M$\rightarrow$N which includes the projectivity π. Moreover, if π is injective, then φ and β are both injective. The homomorphism φ is unique up to conjugacy and β is unique up to multiplication by a unit of B.

A theorem of this kind was established by Ojanguren and Sridharan [6] over commutative rings, but with a much stronger definition of a projectivity than ours. On the other hand, they make no assumption on the stable range of A. They assume that π is bijective; however, the crucial difference from our situation is that they do not restrict the lines of P(M) to be direct summands of M. Sarath and Varadarajan [7] have extended the results in [6] to some non-commutative rings, including von Neumann regular rings and semiprime right Goldie rings with a condition on the right Goldie dimension similar to our stable range condition, but they use the same definition of projectivity as in [6].

In applications of the fundamental theorem of projective geometry over rings to determine the isomorphism of classical groups over rings it has been necessary to use a form of the theorem where lines are direct summands and this appears to be the more natural condition.

In fact, special cases of the theorem above, where A is a commutative local or semilocal ring, can be found in McDonald [5] and James and Weisfeiler [3] and are used there in studying group isomorphisms.

Finding a generalization to other rings, for example polynomial rings, is problem XXIII in [4]. Veldkamp [10] has also established the special case n=3 under the assumption that both SR_2 (A) and SR_2(B) hold.

Veldkamp does not assume our condition on projective frames; it appears to be a consequence of his axioms for distant-preserving homomorphisms which are different from our axioms for a projectiivity.

Following Bass [1], we say the stable range condition $SR_n(A)$ hods if $m \geq n$ and if $(a_1,\ldots,a_m) \in A^m$ is unimodular, then there exist $a'_i = a_i + b_i a_m$ with $b_i \in A$, $1 \leq i \leq m-1$, such that $(a'_1, \ldots, a'_{m-1}) \in A^{m-1}$ is unimodular. For example, if A is semilocal then $SR_2(A)$ holds, and if $A = F[X_1,\ldots,X_k]$ is a polynomial ring in k commuting variables over a field F the SR_{k+2} (A) holds. if $SR_h(A)$ holds, clearly SR_{h+1} (A) holds.

In [9], Veldkamp has given an axiomatic charactarizations of projective plances over rings A satifying $SR_2(A)$. Our result suggests the possibility of giving axiomatic charactarizations of higher dimension projective spaces over more general rings with the aid of stable range conditions. Part of Veldkamp's axiomatization involves neighboring points. Two distinct points are neighbors if they do not generate a line.

Our definition of a projectivity can be rephrased in this terminology.

In the final section of this paper we will give a geometrical characterization of the stable range condition $SR_n(A)$.

2. PROOF OF THE FUNDAMENTAL THEOREM

Let $\Im = \{u_1A, \ldots, u_nA\}$ be a projective frame for P(M) where $\pi u_iA = v_i B$ and $\pi\Im$ is a projective frame for P(N) then $M = u_1 A + \ldots + u_n A$ and $N = v_1 B + \ldots + v_n B$ where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are now bases for M and N, respectively. Since $L = u_1 A + u_2 A$ is a line, clearly $\pi L = v_1B + v_2B$. As $(u_1+u_2)A$ and u_2A also generate the line L, $\pi(u_1 + u_2)A = (v_1b_1 + v_2b_2)B$ for some $b_1, b_2 \in B$. Since $(v_1b_1 + v_2b_2)B$ and v_2B must generate πL, we may assume $b_1 = 1$. Likewise, b_2 is a unit in B and by changing the choice of v_2 in N we have $\pi(u_1 + u_2)A = (v_1 + v_2)B$.

In general, for $2 \leq i \leq n$,

$$\pi(u_1 + u_i)A = (v_1 + v_2)B$$

for any $a \in A$, the points $(u_1 + u_2a)A$ and u_2A generate the line L. Hence $\pi(u_1 + u_2a)A = (v_1 + v_2\varphi(a))B$ for some element $\varphi(a)$ in B. This defines a map $\varphi : A \rightarrow B$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. By essentially the same argument given by Ojanguren and Sridharan [6] it can be shown that φ is a ring homomorphism and

$$\pi(u_1 + u_ia)A = (v_1 + v_i\varphi(a))B$$

for $2 \leq i \leq n$. This part of the proof does not use any assumption on the stable range of A. If $a \in \ker\varphi$, then $(u_1 + u_ia)A$ and u_1A have the same image under the projectivity π. Thus if π is injective, φ is necessarily amonomorphism. Define aφ-semilinear map $\beta : M \rightarrow N$ by $\beta(u_i) = v_i$, $1 \leq i \leq n$. This map will be injective whenever φ is injective. We show next that β induces the projectivity π, that is, if $P = xA$ is a point of $P(M)$, then $\pi P = \beta(x)B$.

Let $x = \Sigma^n u_i a_i$ where $(a,\ldots,a_n) \in A^n$ is unimodular.

when $a_1 = 1$ we can still use the argument of [6,p.313] to show that P has image $\beta(x)$ B. It follows that $\pi(u_i + u_j)A = (v_i + v_j)B$ for any $i \neq j$.

Again, as in [6,p.314], it than follows that $\pi P = \beta(x)$ B for any $x = \Sigma\ u_i a_i$ with at least one confficient $a_k = 1$ the rest of the proof in [6] must , however be changed for it involves rank two modules ("lines ") which need not be direct summand of M.

Let $P = xA$ where $x = \Sigma\ u_i a_i$ with $(a_1,\ldots, a_n) \in A^n$ unimodular. Since the condition SR_n (A) holds, there exist $b_1,\ldots,b_{n-1}$ in A such that $(a'_1,\ldots,a'_{n-1}) \in A^{n-1}$ is unimodular, where $a'_i = a_i + b_i a_n$, $1 \leq i \leq n-1$.

Put $u'_n = u_n - u_1 b_1 - \ldots - u_{n-1} b_{n-1}$ and $v'n = \beta(u'_n)$.

Then $x = u_1 a'_1 + \ldots + u_{n-1} a'_{n-1} + u_n a'_n$. Now change to a new basis $u_1, \ldots, u_{n-1}, u_n$ of M. Since by what has already been established,

$$\pi u'_n A = v'_n B \text{ and } \pi (u_i + u'_n) A = (v_i + v'_n)B,\ 1 \leq i \leq n-1,$$

we can repeat the earlier argument for this new basis and obtain a homomorphrism Ψ: $A \to B$ having analogous properties for this basis as φ does for the original basis. By considering the image of the point $(u_1 a + u'_n)$ A, $a \in A$, in two ways it is easily seen that $\varphi = \psi$. thus in studying the image of P it may now be assumed that $(a_1,\ldots,a_{n-1}) \in A^{n-1}$ is unimodular.

We will now make another change of basis. Since $(a_1,\ldots,a_{n-1})$ is unimodular, there exit $c_1,\ldots,c_{n-1}$ in A such that $c_1a_1,\ldots,c_{n-1}a_{n-1} = 1- a_n$. This time put $u'_i = u_i - u_n c_i$, $1 \le i \le n-1$, and $u'_n = u_n$, so that :

$x = \Sigma^n{}_1 u_i a_i = \Sigma^{n-1}{}_1 u'_i a_i + u'_n$. Then, by what has already been established, $\pi u'_i A = \beta(u'_i) B$, $1 \le i \le n$, and $\pi(u'_i + u'_i)B = \beta(u'_i + u'_j)B$ for $i \neq j$. By repeating the argumefor the new basis $u'_i, \ldots, u'_n$ we obtain a homomorphism ψ: $A \rightarrow B$ having analogous properties for this basis as ϕ did for the earlier basis. By considering the image of the point :

$(u'_i + u'_n a)A$, $a \in A$, in two ways we find $\psi = \varphi$. Since now the u'_n-coefficient of x is one, it follows that $\pi P = \beta(x) B$. Thus β induces π.

Notice, we have also shown that any unimodular element x of M can be expanded to a basis of M (assuming SR_n (A) holds).

Let β' : $M \rightarrow N$ be a second φ'-semilinear map inducing the same project π.

Thus $\beta(x) B = \beta'(x) B$ for all $xA \in P(M)$.

Hence $\beta'(u_i) = \beta(u_i)e = v_i e$, $1 \le i \le n$, where e is a unit of B independent of i.

Then $\beta'(u_1 + u_2 a) = \beta(u_1 + u_2 a)e$ for any $a \in A$. Hence $\varphi'(a) = e^{-1} \varphi(a)e$, so that φ and φ' are conjugate homomorphisms. Now $\beta'(x) = \beta(x) e$ for all $x \in M$, completing the proof

Remark : It is not clear in general when a φ-semilinear mapping β: M $\rightarrow$N induces a projectivity. However, assuming SR_{n-1} (A) holds and β carries at least one basis of M onto a basis of N, then β does induce a projectivity. For now, if the points P_1 and P_2 generate the line L in M, then M = L $\oplus$ U a free module of rank n-2 (see comment two paragraphs above). It is then easily seen that $\beta(P_1)$ B and $\beta(P_2)$ B are point which generate the lineβ(L) B in N.

Note also that even if the projectivity π in the fundamental theorem is bijective, it does not follow that φ is surjective (see [6]).

3.GEOMETRICAL CHARACTERIZATION OF SR_n (A)

Let n = rank M $\geq$ 2. submodule H of M is called a hyperplane if M /H is a free rank one A-module.

Proposition The stable range condition SR_n (A) holds if and only if M has the following property : "for each point P and each hyperplane H of M there exists point Q such that P + Q is a line and H +Q =M".

Proof: Assume first the property holds for M. Let$(a_1,...,a_n)\in A^n$ be unmodular. Let $u_1,....,u_n$ be a basis for M and put $x =\Sigma u_i a_i$,P= xA and H = $u_1.a$ + ...+ u_{n-1}A. Then M/H $\approx u_n$ A, so that H is a hyperplane. By the property there exists a point Q = y A where y = $\Sigma u_i b_i$ such that P + Q is a line and H +Q = M. The condtion H + Q = M forces b_n to be a unit of A and, without loss of generatilty, we may assume b_n = 1.

Since $L = P + Q = xA + yA$ is a line, it is free of rank two, with x,y a basis. Hence $(x-ya_n)A$ is a point, and $(a_1 - b_1a_n, \dots, a_{n-1}-b_{n-1}a_n)$ is unimodular. It follows (using Theorem 1 in [8]) that $SR_n(A)$ holds.

Conversely, assume $SR_n(A)$ holds. Now if $H \subset M$ is a hyperplane, $M = H + R$ where R is a point and H is then free of rank n-1. Hence M has a basis $u_1, \dots, u_n$ where $R = u_n A$ and $H = u_1 A + \dots + u_{n-1} A$.

Let $P = xA$ where $x = \Sigma\, u_i a_i$ with $(a_1, \dots, a_i) \in A^n$ unimodular. Then there exit $b_1, \dots, b_{n-1}$ in A such that $(a_1 - b_1a_n, \dots, a_{n-1}-b_{n-1}a_n)$ is unimodular. Put $Q = yA$ where $y = \Sigma_1^{n-1} u_i b_i + u_n$. Then Q is a piont, $H + Q = M$ and $P + Q = xA + yA = (x-ya_n)A = yA$ is a free rank two direct summand of M (since $x-ya_n \in H$ is unimodular). Thus the sataed property holds.

Remark : The two conditions P + Q is a line and H + Q = M can be restated in terms of neighbor relations (cf. [9]).

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