Sci. J. Fac. Menoufia Univ. Vol. IV (1990). 297 - 307

RESULTS CONCERNING THE CONJUGATE OF A MULTIPARAMETER PROBLEM

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1 - INTRODUCTION

Let X_r, Y_r, $r = 1,\ldots, n$ be *Banach* spaces, and let T_r, V_{rs}, $s = 1,\ldots,n$ be densely defined linear operators in X_r. A multiparameter linear operator

$$L_r(\lambda_\sim) := (T_r + \Sigma^n_{s=1} \lambda_\sim V_{rs}) : X_r \text{------------>} Y_r,\ r = 1, \ldots, n \qquad (1.1)$$

where $\lambda_\sim = (\lambda_1, \ldots, \lambda_n) \in ¢^n$ is an n-tuple of spectral parameters linking the n operators, is known as a multiparameter problem.

In recent years, an abstract spectral theory for multiparameter problems has been developed for a system of equations

$$L_r(\lambda_\sim) f_r = g_r,\ r = 1, \ldots, n \qquad (1.2)$$

where $X_r = Y_r = H_r$, $r = 1,\ldots, n$ are complex *Hilbert* spaces and $L_r(\lambda_\sim)$, $r = 1, \ldots, n$ are self adjoint operators: in particular T_r has been self - adjoint while V_{rs} has been bounded and self adjoint. Given a certain "definiteness condition" on the operators, the spectrum of the multiparameter problem has defined and studied in terms of an associated set of commuting self adjoint operators in the tensor product space $H = H_1 \oplus \ldots \oplus H_n$. For a full account of this work, we site the monograph by Sleeman [8].

However, little work has been done on multiparameter problems involving nonself adjoint operators. In this case, the definition of the spectrum outlined above is inappropriate, and it must be defined more directly in terms of the invertibility of the operators $L_r(\lambda\sim)$, $r = 1, ..., n$ in (1.2) - see [5, 7].

In [5], the notation of an adjoint multiparameter problem in *Hilbert* space is discussed : $(T^* + \Sigma^n_{s=1} \lambda^-_s \; V^*_s)$ is cosidered to be the adjoint of $(T + \Sigma^n_{s=1} \lambda_s \; V_s)$, where * indicates the adjoint of an operator in the usual sense. Some well known properties relating to the spectrum of a linear operator and that of its adjoint are extended to the multiparameter setting under the assumption that

$$(T + \Sigma^n_{s=1} \lambda_s \; V_s)^* = (T^* + \Sigma^n_{s=1} \lambda^-_s \; V^*_s)$$

We note that this assumption is true if V_s, $s = 1, ..., n$ are bounded [2, 6], so that this assumption holds in the self adjoint problems studied in recent years.

Here, we show that this assumption holds under some conditions which allow V_s, $s = 1, ..., n$, to be unbounded. These conditions involve the notations of *Fredholm* operators, relative boundedness and relative compactness of operators. We define these in ◊ 2 and ◊ 3. In 4, we derive our main result concerning the conjugate of a sum of operators and show its application to multiparameter problems.

2 - Relative Boundedness and Relative Compactness

The concepts of relative boundedness and relative compactness of an operator are used extensively in the development of perturbation theory for linear operators [2, 4, 9].

Definition 2.1 Let X, Y be *Banach* spaces. Let T, A be linear operators with domains in X and ranges in Y and $D(T) < D(A)$;

(i) A is said to be relatively bounded with respect to T, or T-bounded if

$$\| Au \| \leq a \| u \| + b \| Tu \| , \quad u \in D(T) \qquad (2, 1)$$

where a and b are nonnegative contstants. The infimum, $b_\circ$, of all possible constants b is caled the T-bound of A.

(ii) A is said to be compact with respect to T, or T-compact, if, for any sequence $\{u_n\} \subset D(T)$, for which $\{u_n\}$ and $\{Tu_n\}$ are bounded, $\{Au_n\}$ contains a convergent subsequence.

The following generalises Theorem IV 1.1 in [3].

Lemma 2.2 Let T, A_s, s=1, ... , n be linear operators form X into Y, and A_s be T-bounded operators,

$$\| A_s u \| \leq a_s \| u \| + b_s \| Tu \| , \; u \in D(T), \; s=1, \dots, n.$$

Then $\sum_{s=1}^{n} A_s$ is T-bounded. If $\sum_{s=1}^{n} b_s < 1$, then $L = T + \sum_{s=1}^{n} A_s$ is closable if if and only if T is colsable. Further L is colosed if and only if T is closed.

Proof: For $u \in D(T) < \bigcap_{s=1}^{n} D(A_s)$

$$\| \sum_{s=1}^{n} A_s u \| \leq \sum_{s=1}^{n} \| A_s u \| \leq \left(\sum_{s=1}^{n} a_s \right) \| u \| + \left(\sum_{s=1}^{n} b_s \right) \| Tu \| \quad (2.2)$$

Now, for $u \in D(T) = D(T)$

$$\| Lu \| = \| Tu + \sum_{s=1}^{n} A_s u \| \leq \| Tu \| + \| \sum_{s=1}^{n} A_s u \|$$

Form (2.2) we get

$$\| Lu \| \leq (\sum_{s=1}^{n} a_s) \| u \| + (1 + \sum_{s=1}^{n} b_s) \| Tu \| \tag{2.3}$$

Also, from (2. 2) we get

$$\| Lu \| \geq \| Tu \| - \| \sum_{s=1}^{n} A_s u \| \geq -(\sum_{s=1}^{n} a_s) \| u \| + (1 - \sum_{s=1}^{n} b_s) \| Tu \|. \tag{2.4}$$

From (2. 3) and (2. 4) we get

$$-(\sum_{s=1}^{n} a_s) \| u \| + (1 - \sum_{s=1}^{n} b_s) \| Tu \| \leq \| Lu \| \leq (\sum_{s=1}^{n} a_s) \| u \| + (1 + \sum_{s=1}^{n} b_s) \| Tu \| . \tag{2.5}$$

Suppose $\{u_n\}$, D (T) is T-convergent sequence i.e. u_n ------------> u and T u_n -----------> v as n -----------> ∞. From the second inequality of (2. 5) we see that $\{u_n\}$ is also L-convergent. Similarly we see from the first inequality of (2. 5) that an L-convergent sequence $\{u_n\}$ is T-convergent. If $\{u_n\}$ is L-convergent to zero i.e. u_n ---------> 0 and L u_n -----------> v as n -----------> ∞, it is T-convergent to zero i.e. u_n -------> 0 and T u_n --------> w as n --------> ∞. If T s closable this implies T u_n ----------> 0 and it follows from the second inequality of (2. 5) that L u_n ----------> 0, which shows that L is closable Similarly, T is closable if L is closable.

That L is closed if and only if T is closed follows as ablve.

<u>Corollary 2. 3</u> Let T, A_s, s=1, ... , n be as in Lemma 2. 2 where $\sum_{s=1}^{n} b_s < 1$.

Then A_{s+1} is L_s-bounded where $L_s = T + \sum_{i=1}^{s} A_i$, $1 \le s \le n-1$, with L_s-bound smaller than one

Proof: Since $Tu = L_s u - \sum_{i=1}^{s} A_i u$, $u \in D(T)$. Then

$$\| Tu \| \le \| L_s u \| + \sum_{s=1}^{n} \| A_i u \| \le \| L_s u \| + \sum_{s=1}^{n} a_i \| u \| + \sum_{s=1}^{n} b_i \| Tu \|$$

i. e. $\| Tu \| \le (1 - \sum_{s=1}^{n} b_i)^{-1} (\sum_{s=1}^{n} a_i \| u \| + \| L_s u \|)$.

But

$$\| A_{s+1} u \| \le a_{s+1} \| u \| + b_{s+1} \| Tu \|$$

hence

$$\| A_{s+1} u \| \le \alpha_{s+1} \| u \| + \beta_{s+1} \| L_s u \|$$

where $\alpha_{s+1} = (1 - \sum_{s=1}^{n} b_i)^{-1} (a_{s+1} + \sum_{s=1}^{n} \Delta_{i(s+1)})$; $\Delta_{i(s+1)} = \begin{vmatrix} a_i & b_i \\ a_{s+1} & b_{s+1} \end{vmatrix}$, and

$$\beta_{s+1} = (1 - \sum_{s=1}^{n} b_i)^{-1} b_{s+1}.$$

Since $\sum_{s=1}^{n} b_i < 1 => \sum_{s=1}^{n} b_i + b_{s+1} < 1 => b_{s+1} < 1 - \sum_{s=1}^{n} b_i => \beta_{s+1} < 1.$

Therefore A_{s+1} is L_s-bounded with L_s-bound smaller than one.

For T-compact operators we have the following result which is generalisation of Theorem IV 1.11 of [4].

Lemma 2.4 Let T, A, B be linear operators from X into Y, where A and B are T-compact operators. If T is closable, then S=T+A is closable and B is S-compact. Further, S is closed if T is closed.

Proof: Assume that $\{u_n\}$ and $\{Su_n\}$ are bounded sequence in X: we require to show that $\{Bu_n\}$contains a convergent subsequence. Since B is T-compact, this follows if we can show that $\{Tu_n\}$ contains a bounded subsequence. Therefore, suppose this false, so that $\| Tu_n \|$ -----------> ∞. Set $v_n = v_n / \| Tu_n \|$. Then, as n -----------> ∞

(i) v_n ---------> 0, (ii) Sv_n-------> 0 (iii) $\| Tv_n \| = 1$ (2.6)

Since A is T-conpact, $\{Av_n\}$ contains a convergent subsequence. Thus, replacing $\{u_n\}$ by a suitable subsequence, we may assume

Av_n -----------> w. (2.7)

Then, using (2.6) (ii) and (2.7)

$Tv_n = Sv_n - Av_n$ ----------> -w. (2.6)

Now, since T is closabel (2.6) (i) and (2.8) imply that w=0. However, this contradicts the fact (-w) = $\lim Tv_n$ and $\| Tv_n \| = 1$, for all n. Therefore the supposition that $\{Tv_n\}$ does not contain a bounded subsequence is false.

Thus, again replacing $\{u_n\}$ by a suitable subsequence so that $\{u_n\}$ and $\{Tu_n\}$ are bounded, we deduce that $\{Bu_n\}$ contains a convergent subsequence since B is T-compact. Thus, B is S-combact.

That S is closable, and S is closed if T is closed follow directly from [4, Theorem IV 1.11].

◊ 3 Fredholm Operators

Definition 3.1 A linear operator T from a *Banach* space X into a *Banach* space Y is said to be a *Fredholm* operator if T is closed, its range $\mathcal{R}(T)$ is closed, and its nullity $\alpha(T) = \dim N(T)$ and deficiency $\beta(T) = \operatorname{codim} \mathcal{R}(T)$ are both finte.

The set of all *Fredholm* operators from X into Y is denoted by $\phi(X, Y)$. The following properties of *Fredholm* operators are well known [2, 4].

P1: If T is a densely defined then $T \in \phi(X, Y)$ if and only if $T' \in \phi(X', Y')$ where T' is the conjugate operator of T.

P2: If $T \in \phi(X, Y)$ and A is T-bounded with $a + \gamma(T)\, b < \gamma(T)$, where $\gamma(T)$ is the minimum modulus of T, then $T + A \in \phi(X, Y)$.

P3: If $T \in \phi(X, Y)$ and A is T-compact then $T + A \in \phi(X, Y)$.

◊ 4 The Conjugate of a Sum of Operators

If T and A are densely defined operators then it is well known that $(T+A)' \geq T' + A'$. The following results provide sufficient conditions for equality:

Theorem 4.1 Let T and A be densely defined linear operators from a *Banach* space X into a *Banach* space Y,

(i) If T is closed, $D(T) \leq D(A)$, $D(T') \leq D(A')$, and there exist constants a, $b \in R$, $b < 1$ such that

$$\| Au \| \leq a \| u \| + b \| Tu \| \quad , \quad u \in D(T);$$
$$\| A' f \| \leq a \| f \| + b \| T' f \| \quad , \quad f \in D(T')$$

Then, $(T+A)' = T'+A'$.

(ii) If $T \in \phi(X, Y)$ A is T-combact and A' is T'-combact, then $(T + A)' = T' +A'$.

Proof: We refer to Hess & Kato [3] for proving (i) and to Beals [1] or Goldberg [2 p. 124] for proving (ii).

The results of ◊ 2 allow us to extend these results.

Theorem 4.2 Let T, A_s, $s=1, ..., n$ be densely defined operators from X into Y and T be closed. If

(i) $D(T) < \bigcap_{s=1}^{n} D(A_s)$ and $D(T') < \bigcap_{s=1}^{n} D(A_s')$;

(ii) there exist constants $a_s, b_s \in R$, $s=1, ... , n$, $\sum_{s=1}^{n} b_s < 1$ such that

$$\| A_s u \| \leq a_s \| u \| + b_s \| Tu \| \quad , \quad u \in D(T);$$
$$\| A_s' f \| \leq a_s \| f \| + b_s \| T' f \| \quad , \quad f \in D(T')$$

$$\text{then } (T+\sum_{s=1}^{n} A_s)' = T' + \sum_{s=1}^{n} A_s' \tag{4.1}$$

Proof: The result is true for n=1 {it becomes Theorem 4.1 (i)}. Assume that (4.1) holds for n=k. Let $L_K = T+ \sum_{s=1}^{K} A_s$, so that $L_K' = T'+ \sum_{s=1}^{K} A_s'$, By Corollary

2.3, A_{K+1} is L_k-bounded with L_K-bound smaller than one and also $A_{K+1}`$ is $L_K`$ -bounded with the same L_K-bound as L_k-bound. From Lemma 2.2, L_K is closed operator. Therefore, by Theorem 4.1

$$(L_K + A_{K+1})` = L_K` + A_{K+1}`$$

i.e $$(T+\sum_{s=1}^{K+1} A_s)` = T` + \sum_{s=1}^{K+1} A_s` .$$

Thus, the result is proved by induction for any finite n.

Now we consider the generalisation of Theorem 4.1 (ii) when the operators A_s, s=1, ..., n are T-compact. The technique of the proof is the same as Theorem 4.2. We omit the proof.

Theorm 4.3 Let $T \in \phi(X, Y)$ with D (T) dense in X. If $A_s : X \geq D(A_s)$ --------> Y, s=1, ..., n are T-compact and $A_s`$, s=1, ...., n are $T`$-compact, then

$$(T+\sum_{s=1}^{n} A_s)` = T` + \sum_{s=1}^{n} A_s` .$$

Corollary 4.4 Let T and A_s as in Theorem 4.3, then

$$(T+\sum_{s=1}^{n} \lambda_s A_s)` = T` + \sum_{s=1}^{n} \lambda_s A_s` , \text{ for all } (\lambda_s, ..., \lambda_s) \in ¢^n .$$

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INTRODUCTION

n-tuple of spectral parameters linking the n

Sleeman [8]. - see [5, 7]. In [5], [2.6], so that

[2, 4, 9]. T-compact, $\subset D$ $u \in$ $\subset D$ T

$\sum // A_i u // \leq // L_s u // + \sum a_i // u // + \sum b_i) // Tu //$

$\sum b_i + b_{s+1} < 1 \Rightarrow b_{s+1} < 1 - \sum b_i \Rightarrow \beta_{s+1} < 1.$

T, A, B

(i) v_n ---------> 0, (ii) Sv_n--------> 0 (iii) $// Tv_n // = 1$ (2.6)

Then, $\supseteq$ $\supseteq$ $\supseteq$ $\supset$ $\supset$ $\supset$ $\supseteq$ $\supseteq$

2.3, A_{K+1} is L_k -bounded with L_K -bound smaller than one and also A_{K+1}\` is $L_K$\` -bounded with the same L_K -bound as L_k -bound. From Lemma 2.2, L_K is closed operator. Therefore, by Theorem 4.1

$$(T + \sum \lambda_s A_s)` = T` + \sum \lambda_s A_s` , \text{ for all } (\lambda_s, ..., \lambda_s) \in ¢^n .$$