Date: 25 / 9 / 2013 Time: 3 hours Full mark: 60 marks	M. Sc. Final Exam Sept. 2013 Abstract Algebra Code: 8500	Mansoura University Faculty of Engineering Math. and Engineering Phys. Dept.

Prob. (1) [12 pt.]

(a) [4 pt.] Define each of the following:

i) Invariant subgroup, ii) Homomorphism,

iii) Maximal element, iv) Quotient group.

(b) [4 pt.] State the difference between each of the following:

i) Right coset and left coset, ii) Into mapping and onto mapping,

iii) Proper and improper subgroups, iv) Homomorphism and isomorphism.

(c) [4 pt.] State that each of the following statements is true or false and correct the false statements:

i) If α is a mapping of a set S onto a set T, then α has a unique inverse. ( )

ii) A elation R on a set S is called reflexive if whenever aRb then bRa. ( )

iii) An equivalence relation R on a set S effects a partition of S, and conversely, a partition of S defines an equivalence relation on the set S. ( )

(iv) Homomorphic image of any cyclic group is cyclic. ( )

Prob. (2) [12 pt.]

(a) [3 pt.] Prove that; If α is one to one mapping of a set S onto a set T then α has a unique inverse and conversely.

(b) [3 pt.] Prove that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

(c) [3 pt.] Prove that: $A-(B\cup C)=(A-B)\cap(A-C)$.

(d) [3 pt.] Prove that:

i) $x \to x+2$ is a mapping of $\mathbb{N}$ into, but not onto, $\mathbb{N}$.

ii) $x \to 3x-2$ is a one-to-one mapping of $\mathbb{Q}$ onto $\mathbb{Q}$.

iii) $x \to x^3-3x^2-x$ is a mapping of $\mathbb{R}$ onto $\mathbb{R}$ but is not one-to-one.

Prob. (3) [14 pt.]

(a) [2 pt.] Show that " is congruent to" on the set T of all triangles in a plane is an equivalence relation.

(b) [2 pt.] Prove that if $[a]\cap[b]\neq\varnothing$, then $[a]=[b]$.

(c) [3 pt.] Prove that: The identity element, if one exist, with respect to a binary operation $\circ$ on a set S is unique.

(d) [4 pt.] Express in cyclic notation on 5 symbols:

i) the product $(23)\circ(13)(245)$ and $(13)(245)\circ(23)$,

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ii) the inverse of (23) and (13)(245).

(e) [3 pt.] show that multiplication is a binary operation on $S=\{1, -1, i, -i\}$ where $i=\sqrt{-1}$.

Prob. (4) [14 pt.]

(a) [3 pt.] Show that $\mathcal{G}$, the additive group Z_4, is isomorphic to $\mathcal{G}'$, the multiplicative group of none-zero elements of Z_5.

(b) [3 pt.] Does the set of non-zero residue classes modulo 4 form a group with respect to addition? with respect to multiplication?

(c) [2 pt.] Prove that when $a, b \in \mathcal{G}$, each of the equations $a \circ x = b$ and $y \circ a = b$ has a unique solution.

(d) [3 pt.] Prove that a non empty subset $\mathcal{G}'$ of a group $\mathcal{G}$ is subgroup of $\mathcal{G}$ if and only if, for all $a, b \in \mathcal{G}'$ and $a^{-1} \circ b \in \mathcal{G}'$.

(e) [3 pt.] Prove that: In a homomorphism between two groups $\mathcal{G}$ and $\mathcal{G}'$, their identity element correspond, and if $x \in \mathcal{G}$ and $x' \in \mathcal{G}'$ correspond so also do their inverses.

Prob. (5) [12 pt.]

(a) [2 pt.] Prove that: If R is a ring with zero element z, then for all $a \in R$, $a \cdot \mathbf{z} = \mathbf{z} \cdot a = \mathbf{z}$.

(b) [3 pt.] Prove that: if p is an arbitrary element of a commutative ring R, then $P=\{p.r : r \in R\}$ is an ideal in R.

(c) [4 pt.] Prove that; the set $M=\{(a, b, c, d) : a, b, c, d \in \mathbb{Q}\}$, with addition and multiplication defined by

$$(a, b, c, d)+(e, f, g, h)=(a+e, b+f, c+g, d+h)$$
$$(a, b, c, d).(e, f, g, h)=(ae+bg, af+bh, ce+dg, cf+dh)$$

For all $(a, b, c, d), (e, f, g, h) \in M$ is a ring.

(d) [3 pt.] Prove that; the set $P=\{(a, b, -b, a) : a, b \in \mathbb{Z}\}$, with addition and multiplication defined by

$$(a, b, -b, a)+(c, d, -d, c)=(a+c, b+d, -b-d, a+c)$$
$$(a, b, -b, a).(c, d, -d, c)=(ac-bd, ad+bc, -ad-bc, ac-bd)$$

is a commutative subring of the non-commutative ring M of **problem (5c).**

With all best wishes
Dr. Waleed El-Beshbeeshy