Sci. J. Fac. Sci. Menoufia Univ. Vol. VIII (1994) 127-139

GRAPHS ASSOCIATED WITH NEAT SIMPLICIAL FOLDING

By

E. M. El-Kholy

Department of Mathematics, Faculty of Science,

Tanta University, Tanta, Egypt.

ABSTRACT

In this paper I have constructed a graph associated to the n-simplexes of an n-simplicial complex and its simplicaial folding in a natural way. This graph is connected and vertex transitive for simplicaial neat folding.

By using this graph I obtained the necessary and sufficient condition for a simplicial map to be a simplicial neat folding.

INTRODUCTION

Let K and L be simplicial complexes, a simplicial map ϕ from K to L, $\phi : K \rightarrow L$ is a simplicial folding if for every i and all $\sigma \in K^{(i)}$, $\phi(\sigma) \in L^{(i)}$, i.e. a simplicial ma p ϕ is a simplicial folding if ϕ maps i-simplexes to i-simplexes,[2]. In the case of oriented simplicial complexes it is also supposed the ϕ maps oriented i-simplexes of K to i-simplexes of L but of the same orientation [1].

Let K and L be simplicial complexes of the same dimension n. A simplicial folding $f : K \rightarrow L$ is a simplicial neat folding if and only if there is a siplicial subdivision on L for which $L^{(n)}$ consists of the single n-simplex, Int L, [3].

2. The Graph of a Simplicial Folding

Let ϕ: $K \rightarrow L$ be a simplicial folding. By using the simplicial subdivision of K we show in the following that there is a graph G_ϕ associated to the n-simplexes of K and the simplicial folding ϕ in a natural way.

In fact the vertices of G_ϕ are just the n-simplexes of K and if σ and σ' are distinct n-simplexes of K such that $\phi(\sigma) = \phi(\sigma')$, then there exists an edge E with end points σ and σ'.

The graph G_ϕ can be realized as a graph G_ϕ embedded in R^3, as follows: For each n-simplex σchoose any point $v \in \sigma$. If $\sigma, \sigma' \in K^{(n)}$

are end points of an edge E, then we can join v to v' by an arc e in R^3, [4]. The correspondence $\sigma \to v$, $E \to e$ is trivially a graph isomorphism from G_ϕ to G_ϕ. This construction has a greater significance in the case of simplicial neat foldings as we show in this paper.

It shoul be noted that the graph G_ϕ neither has more than one edge joining a gien pair of vertices, nor an edge joining a vertex into itself.

2.1 Theroem:

The graph G_ϕ is disconnected unless ϕ is a simplicial neat folding.

Proof:

Let σ and γ be distinct n-simplexes of $K^{(n)}$ and let $\sigma \sim \gamma_\phi$ measn $\phi(\sigma) = \phi(\gamma)$. It is clear that the relation ~ is an equivalence relation. Hence the quotient set $K^{(n)} /{\sim} = \{[\sigma], \sigma \in K^{(n)}\}$ is a partition on $K^{(n)}$ where $[\sigma]$ is the equivalence class of any n-simplex σ. It follows that G_ϕ has more than one component otherwise all the n-simplex of k will be mapped to the same n-simplex of L which in fact is the case of simplicial neat folding. In the last case there will be a unique equivalence class $[\sigma]$ and hence hence the graph G_ϕ will be connected.

From the above theorem we see that the number of components of G_ϕ is the same as the number of equivalence classes.

I now explore the behaviour of the graph G_ϕ on these equivalence classes.

2.2 Theorem:

Let ϕ be a simplicial folding from K to L, then each component of G_ϕ is a vertex transitive on itself.

Proof:

As we saw by theorem (2.1) ϕ defines a partition $\{[\sigma],\sigma \in K^{(n)}\}$ on the set of n-simplex $K^{(n)}$ in K. Each equivalence class represents a component of $G\phi$. Now, consider one of the components G_ϕ^i, with say r vertices, i.e $|V(G_\phi^i)| = r$. Each vertex in G_ϕ^i is adjacent to the other vertices in the component; then any permutation of the set $V(G_\phi^i)$ is an automorphism of G_ϕ^i. Thus the set of all permutations (automorphisms) form a group which is the symmetric group S_r acting on the set $V(G_\phi^i)$. The orbit of any $\sigma \in V(G_\phi^i)$ under S_r is the whole set $V(G_\phi^i)$ i.e. $V(G_\phi^i)$ has a single orbit and hence the automorphis group S_r is transitive on $V(G_\phi^i)$.

By using the above theorem I have the following results for a simplicial neat folding $\phi : K \rightarrow L$.

(i) The symmetric group S_r, $r = |K^{(n)}|$ acts transitively on the graph G_ϕ

(ii) G_ϕ is a vertex transitive.

From the above results, I conclude that the graph G_ϕ of a simplicial neat folding is a complete graph.

3. The Simplicial Neat Folding And Its Associated Graph.

I first show that it is suficient for a simplicial map to map n-simplex to n-simplexes be a simplicial folding, we use homogeneously n-dimensional simplicial complexes (i.e., every simplex is a face of some n-simplex of the complex.) for this purpose to avoid such a complex shown in Figure (1), [1].

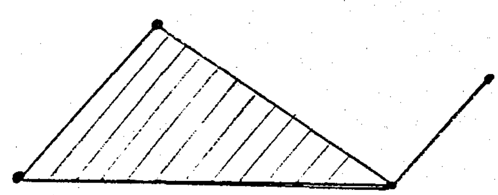


Fig. (1)

3.1 Theorem:

Let K, L-be homgeneously n-dimensional simplicial complexes. A simplicial map ϕ from K to L is a simplicial folding if and only if ϕ maps each n-simplex in K to an n-simplex in L.

Proof:

The necessity condition is obvious. For the converse we have to show that the given condition is suficient for a simplicial map to map each i-simplex of K to an i-simplex of L and hence is a simplicaial folding.

Suppose $\gamma = (\omega^o \omega' \dots \omega^{n-1})$ be an $(n$-1)- simplex of K such that σ (γ) is not an $(n$-1)- simplex of L, i.e., $(\phi\omega^o \phi\omega' \cdots \phi\omega^{n-1})$ has at least two identical vertices. Now, since K is homogeneous, then γ is a face of an n-simplex of K, say $(\omega^o \omega' \dots \omega^{n-1} v)$. But $\phi(\omega^o \omega' \dots \omega^{n-1} v) = (\phi\omega^o \phi\omega' \cdots \phi\omega^{n-1} \phi v)$ is not an n-simples of L and hence contradicts the gien condition. By the same way we can show that ϕ maps each i-simplex of K to an *i-simplex* of L for $i = 1,2,3,\dots,n$-2.

From the above theorem a simplicial map ϕ from K to L is a simplicial neat folding if and only if ϕ maps each n-simplex of K to the single n-simplex , Int L.

3.2 Theorem:

A simplicial map ϕ of homogeneously n-dimensional simplicial complexes is a simplicial neat foldin if and only if the graph G_ϕ is a complete grah.

Proof:

If $\phi : K \rightarrow L$ is a simplicial neat folding then the graph G_ϕ is a complete graph. conversely consider a simplicial map $\phi : K \rightarrow L$. Construct a graph G in the same manner as that mentioned before but noting that ϕ is just a simplicial map. A gain there would be an edge between two vertices $\sigma, \sigma' \in K^{(n)}$ if $\phi(\sigma) = \phi(\sigma')$. Now $\phi(K) = L$ is a simplecial complex of dimension n, then there would exist at least one vertex v in G. Since the graph G is complete, then each other vertex in G will be adjacent to v, i.e., all the n-simplixes of K will be mapped to the same n-simplex in the image of ϕ. From theorem (3.1) it follows that ϕ is a simplicial neat folding.

3.3 Examples:

(a) Let K be a simplicial complex such that $|K|$ is the upper hemisphere with the simplicial subdivisions shown in Figure (2).

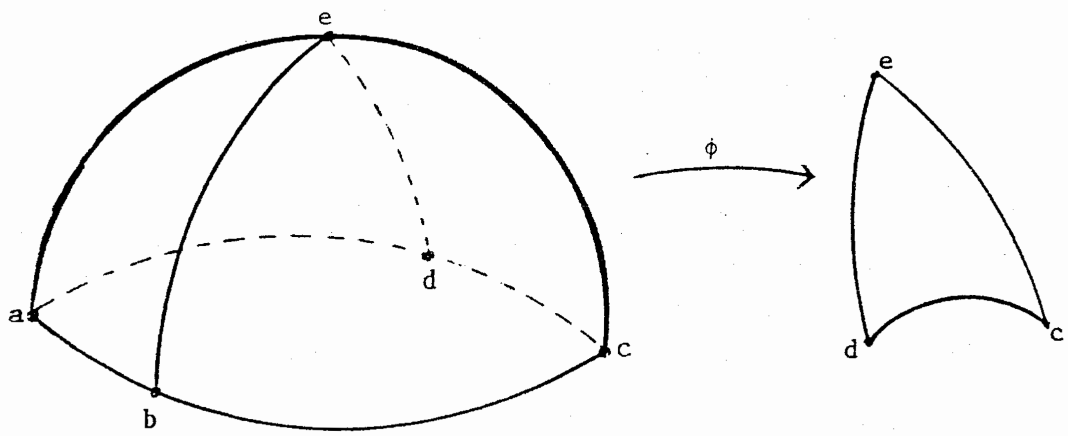
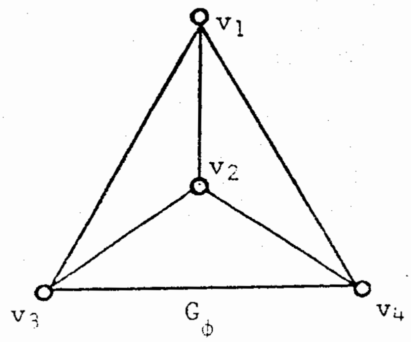


Fig. (2)

Let $\phi : K \to K$ be given by,

$$\phi\,(a,b,c,d,e) = (c,d,c,d,e).$$

Then ϕ is a simplicial neat folding and the graph G_ϕ is a complete graph.



Here v_1, v_2, v_3, v_4 are the 2-simplexes *(aed), (dec), (ceb), (bea).*

(b) Let K and L be simplicial complexes such that $|K|$ is a tetrahedron and L is a 2-simplex [uvw] see Figure (3). Consider a simplicial map $\phi: K \rightarrow L$ defined as follows:

$$\phi(a,b,c,d) = (u,v,w,u)$$

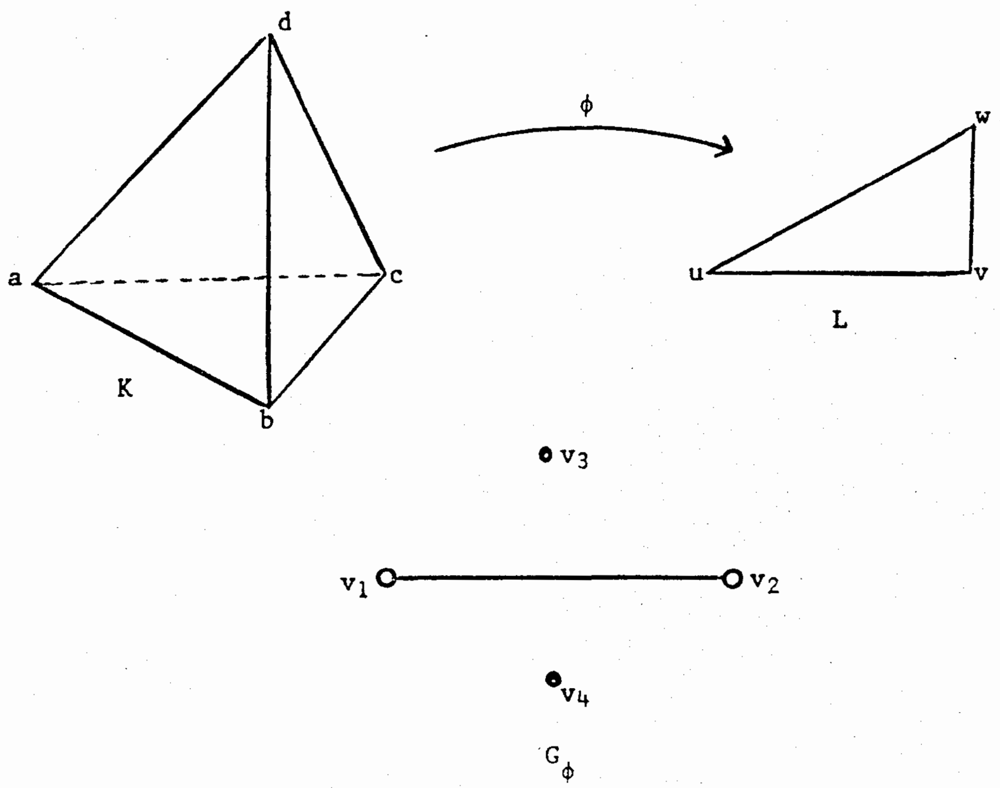


Fig.(3)

It is clear that G_ϕ has four vertices but only one edge and hence is not a complete graph. Thus ϕ is not a simplicial neat folding. This can be easily seen since the 2-simplexes (abd), (acd) are mapped to the 1-simplexes (uv) and (uw) respectively.

We know that for simplicial foldings the graph G_ϕ consists of components each of which is complete. In the following example we show that the converse is not true.

(c) Let K be a simplicial complex such that $|K|$ is homeomorphic to a disc with the triangulation shown in Figure (4).

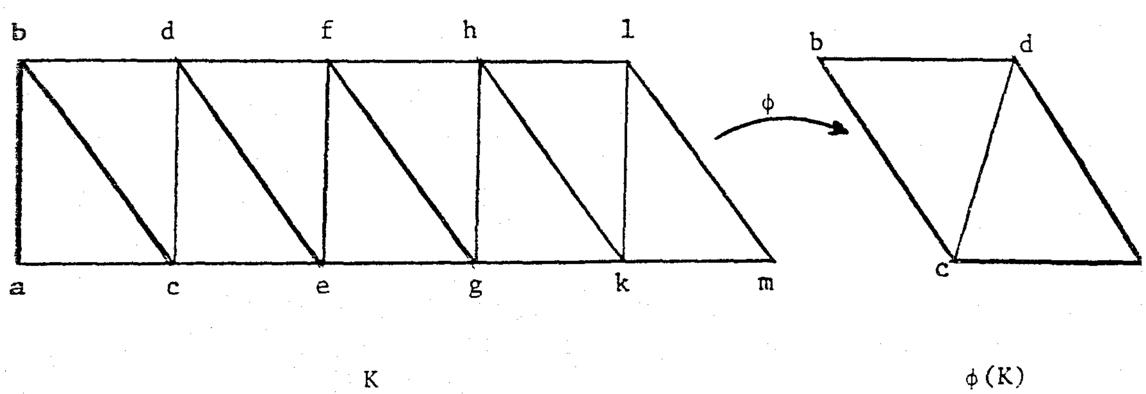
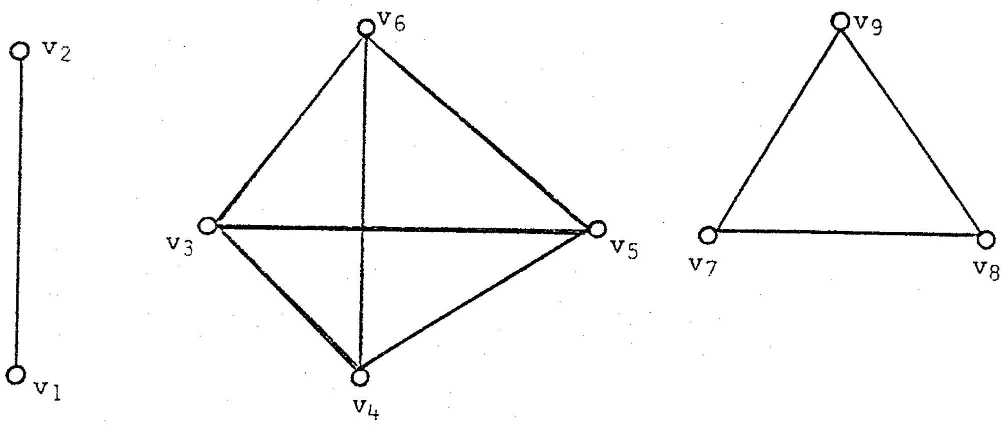


Fig.(4)

Let $\phi : K \to K$ be a simplicial map defined as follows:

$$\phi(a, b, c, d, e, f, g, h, k, l, m) = (d, b, c, d, e, c, d, e, d, e, d)$$

The graph G_ϕ has three components; each is complete, but ϕ is not a simplicial folding since the images of the 2-simplexes (*ghk*), (*hkl*),



G_ϕ

(*klm*) are the 1-simplex (*de*).

REFERENCES

[1] E. El-Kholy and H. A. Al-Khursani, Folding of CW-complexes, Jour. Inst. Mat. & Comp. Sci. (Mat. Ser.) Vol. 4, No. 1, pp. 41-48, 1991.

[2] E. El-Kholy and H. A. Al-Khursani, Folding of simplicial complexes, Jour. Inst. Math. & Comp. Sci. (Math. Ser.) 1992. In Press.

[3] E. El-Kholy and M. El-Ghoul, Simplicial foldings, Jour. Fac. Education, No. 7, Part II, pp. 127-139, 1984.

[4] S. A. Robertson and E. El-Kholy, Topological folding, commun, Fac. Sci. Univ. Ank. Series A_I V. 35, pp. 101-107, 1986.

المخططات المرتبيطة مع الطى الصافى المبسط

إنتصار محمد الخولى
قسم الرياضيات - كلية العلوم - جامعة طنطا
طنطا - مصـــر

فى هذا البحث قمت بإنشاء مخطـط مرتبـط مـع المبسـطات ذوات البعد النونى وذلك للتراكيب المبسـطة النونيـة المطويـة طـى مركـب وذلـك بطريقـة طبيعية.

هذا المخطط مترابط وله تأثير انتقالى على الرؤوس فـى حالـة الطـى الصافى.

باسـتخدام هـذا المخطـط اسـتطعت أن أحصــل علــى الشــرط الكــافى والضرورى للدالة المبسطة حتى تصبح دالة طى صافى مبسط.