Mansoura Engineering Journal, (MEJ), Vol. 36, No. 1, March 2011 E. 1

MODELING INERTIAL SENSOR ERRORS USING ALLAN VARIANCE

نمذجة حساس خطأ القصور باستعمال متغير آلان

A. S. Kraza [1], M. S. Bayoumi [2], G. M. El-Bayoumi [3], S. D. Hassan [4]

مستخلص

نظم الملاحة بالقصور الذاتي توفر دقة عالية للموضع، السرعة، و الإتجاه (بارامترات الملاحة) على مدى فترات زمنية قصيرة، لكن هذه الدقة سرعان ما تتراجع مع مررور الوقت بسبب أخطاء مجسات القصور الذاتي. لرفع كفاءة الملاحة بالقصور الذاتي لابد من تخمين أخطاء المجسات وإزالتها قبل استخدامها في حساب بارامترات الملاحة، ولهذا يعتبر التمثيل الرياضي لأخطاء المجسات ضرورة حتمية. يعد متغير آلان من الطرق البسيطة والفعالة المستخدمة في التمثيل الرياضى لأخطاء المجسات العشوائية وذلك بتمثيل جذر متوسط مربع القيمة لإنحراف الخطأ العشوائي كدالة من متوسط الوقت. متغير آلان يمدنا بمعلومات عن نوع وقيمة الأخطاء العشوائية المختلفة.

تستخدم هذه الورقة تقنية متغير آلان لتحليل وتمثيل الأنواع المختلفة للأخطاء العشوائية الموجودة بقياسات مجسات القصور الذاتي. النموذج الرياضي العشوائي المستنتج لأخطاء المجسات سوف يتم إضافته للنموذج الرياضي لأخطاء نظام الملاحة بالقصور الذاتي لبناء نظام ملاحة متكامل بعد التأكد من صحة النموذج الرياضي العشوائى. وأخيرا، تعرض هذه الورقة العلمية نتائج الإختبارات والتحقق من صحة النموذج الرياضي للأخطاء العشوائية للمجسات.

ABSTRACT

Inertial navigation systems (INS) can provide high-accuracy position, velocity, and attitude information over short time periods. However, their accuracy rapidly degrades with time due to inertial sensor errors. To damp down the error growth, the INS sensor errors should be properly estimated and compensated before the inertial data are involved in the navigation computation. Therefore, appropriate modeling of the INS sensor errors is a necessity. Allan Variance (AV) is a simple and efficient method for modeling and verifying these errors by representing the root mean square (RMS) random drift error as a function of averaging time. Allan variance can provide information on the types and magnitude of the various error terms. This paper uses the AV technique to analyze and model different types of random errors residing in the measurements of Micro Electro Mechanical System (MEMS) based inertial sensors. The derived stochastic error model will be further included in the INS error model for integrated navigation system, once the correctness of the model is verified. Finally, the paper presents the test results and model validation.

KEYWORDS: Stochastic modeling, Allan variance (AV), Inertial sensor errors, INS, MEMS.

1- Ph. D Research student, Dept. of Aerospace Engineering, Faculty of Engineering, Cairo University.

2- Assistant Professor, Dept. of Aerospace Engineering, Faculty of Engineering, Cairo University.

3- Professor, Dr., Dept. of Aerospace Engineering, Faculty of Engineering, Cairo University.

4- Professor, Dr., Head of the navigation and control research group, Dept. of Aerospace Engineering, Cairo University.

Accepted March 20, 2011.

1. INTRODUCTION

An Inertial Navigation System (INS) is a system that calculates the position, velocity and attitude of a vehicle with short-term stability due to the noise characteristics of its inertial sensors. Inertial sensors inherently contain errors in their output. . The errors of inertial sensors are classified into deterministic errors and random errors. In general, deterministic errors are estimated by calibration and therefore, they can be removed from the raw observations. On the other hand, random errors of inertial sensors are a mixture of several basic errors such as white noise, random bias, exponentially correlated (Markov) noise, rate random walk and rate ramp. Therefore, the errors must be identified and modeled before engaging in an inertial system.

The frequency-domain approach for modeling noise by using the power spectral density (PSD) to estimate the transfer functions is straightforward but difficult for non-system analysis to understand. The Allan variance (AV) method is known to be simple and reliable in identifying a model from a signal of mixed noises. Allan variance is a time domain analysis technique originally developed to study the frequency stability of oscillators [2, 4]. It can provide information about the types and magnitude of the errors residing in the measurements.

This paper applies the AV to characterize MEMS based inertial sensor errors through identification of the error statistics. The procedure is as follows: The characteristic curve is first obtained by applying the AV algorithm to the entire data. The curve is then measured to determine the types and magnitudes of certain random errors possibly residing in the data. Finally, the random errors are identified and modeled.

The paper is organized as follows; stochastic modeling of different random errors are discussed first, mathematical definition of the power spectral density (PSD) and the AV and their relationship are summarized. Using this relationship, the behavior of the characteristic curve for a number of prominent noise terms is determined. The test results are presented and verified.

2. STOCHASTIC ERRORS

As the name suggests, the errors are random in nature and there is no repeatability. Even if an experiment with the same instrument and under same conditions was repeated, the random errors that encountered will turn out to be different in different cases. The random errors typically consist of white noise, random bias, random walk, Gauss Markov, etc. Random errors should be modeled stochastically. In stochastic modeling, a model is hypothesized which, as though excited by white noise has the same output characteristics as the unit under test [11].

The most typical stochastic processes used to model the dynamics of the inertial sensor errors are:

2.1 White Noise

White noise $w(t)$ is defined as a stationary random process having a constant spectral density function (q). A number of random processes can be generated by passing white noise through a suitable filter [9].

2.2 Random Constant Model

The random constant (RC) is a non-dynamic quantity with fixed but random amplitude; its continuous and discrete expressions are given respectively by [11].

$$\dot{x}(t) = 0, \qquad x(0) = a_0 \\ x_{k+1} = x_k \tag{1}$$

The initial condition a_0 is a random variable whose distribution is presumed to be known. A random constant is an appropriate model for turn-on to turn-on biases of inertial sensors as these biases can change from one trajectory to another but remains fixed during the specific trajectory.

2.3 Random Walk Model

Random walk (RW), also called Wiener process, is defined as the integral of the white noise with initial conditions zero. The continuous analog of random walk is the output of an integrator driven with white noise. A RW can be described through the following differential equation in continuous time domain [11]:

$$\dot{x}(t) = w(t) \tag{2}$$

From this equation, it can be seen that RW can be generated by integrating an uncorrelated random sequence $w(t)$. In discrete time, the process can be described through the following equation [9]:

$$X_k = X_{k-1} + W_k \tag{3}$$

So, the discrete-time RW model can take the form of Eq. (3). Also, Allan variance analysis can be used to estimate the variance of the driven noise w_k (see section (4) for Allan variance details).

2.4 Gauss Markov Model

Gauss-Markov processes (GMP) are stationary processes that have exponential autocorrelation functions. The GM process is important because it is able to represent a large number of physical processes with reasonable accuracy and has a relatively simple mathematical formulation [9]. A stationary Gaussian process that has an exponentially decaying autocorrelation is called first-order GM process. For a random process x(t) with zero mean, mean squared error σ^2, and correlation time T_c, the first-order GM model is described by the following continuous-time equation [9]:

$$\dot{x}(t) = -\frac{1}{T_c} x(t) + w(t) \tag{4}$$

The autocorrelation function (see Fig. 1) of the first-order GM model is given by [9]:

$$R(\tau) = E[x(t)x(t+\tau)] = \sigma^2 e^{-|\tau|/T_c} \tag{5}$$

Where (τ) is the time shift, T_C is the correlation time, and σ^2 is the variance at zero time shift ($\tau = 0$). From Eq. (4) two parameters namely, T_C (correlation time) and σ_w^2 (driven noise variance), are required to describe a GM process

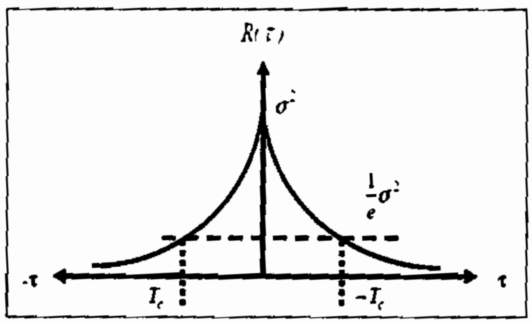

Fig. 1 The autocorrelation function of the first-order Gauss-Markov process.

The first-order GM process in discrete time can be written as [10]:

$$X_k = e^{-\Delta t/T_c} X_{k-1} + W_k \tag{6}$$

And the associated variance can be given by [10]:

$$\sigma_{W_k}^2 = \sigma_{GMP}^2 \left(1 - e^{-2\Delta t/T_c}\right) \tag{7}$$

So, the discrete-time first-order GM model can be applied using Eq. (6) and the variance of the driven noise W_k is given by Eq. (7). The first-order GM process is one of the most commonly applied shaping filters in integrated navigation systems because the bounded uncertainty characteristic of GM process makes it the best model for slowly varying sensor errors such as residual bias [10].

3. POWER SPECTRAL DENSITY (PSD)

The PSD is the most commonly used in representation of the spectral decomposition of a time series. It is a powerful tool for analyzing or characterizing data and stochastic modeling. For a stationary process, the basic relationships between the two-sided PSD and autocorrelation function of a signal are Fourier transform pairs [11], as shown in the equations below:

$$S(w) = \int_{-\infty}^{\infty} K(\tau)\, e^{-jw\tau}\, d\tau \tag{8}$$

The transfer-function form of the stochastic model may be directly estimated from the PSD of the output data (on the assumption of an equivalent white-noise driving function). For linear systems, the output PSD is the product of the input PSD and the magnitude square of the system transfer function [11].

$$S_{output}(w) = G(jw)G(-jw)S_{input}(w) \tag{9}$$

Where

$G(jw)$ System transfer function.

S_{output} Output PSD

S_{input} Input PSD

Thus, for the special case of the white-noise input, (S_{input} is equal to some constant value), the output PSD directly gives the system transfer function.

4. ALLAN VARIANCE

The Allan variance [1, 2, 3] is a method of representing root mean square random drift errors as a function of averaging times. It is simple to compute, and relatively simple to interpret and understand. Its most useful application is in the identification and estimation of random drift coefficient in previously formulated model equations given in section (2). The attractiveness of this method is that, the Allan variance, when plotted in logarithmic scales can discriminate different contributing error sources by simply examining the varying slopes on the Allan plot.

Given the angular rates or acceleration are recorded at a constant time interval t_0, a collection of N data points can be reformed to be $K=N/M$ clusters where M is the number of samples per cluster, as shown in Figure 4.

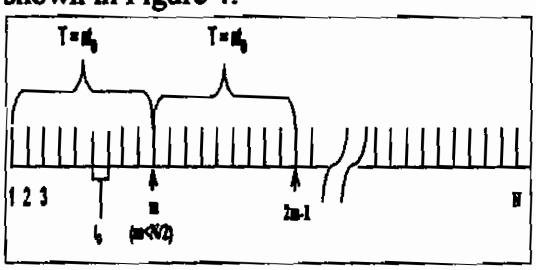

Fig. 4 Schematic diagram of the Data Structure used in the Derivation of Allan Variance

A time T is associated with each cluster which is equal to (nt_0).

Then, the average for each cluster:

$$\overline{\Omega}_k(M)=\sum_{i=1}^{M} D_{ki}; \quad k=1,\dots,K \qquad (10)$$

The Allan Variance from the cluster averages is estimated as follows:

$$\sigma^2(T)=\frac{1}{2(K-1)}\sum_{k=1}^{K-1}\left(\overline{\Omega}_{k+1}(M)-\overline{\Omega}_k(M)\right)^2 \qquad (11)$$

There is a very important relationship between Allan variance and power spectral density (PSD) of a random process [8]:

$$\sigma^2(\mathrm{T})=4\int_0^\infty S_X(f)\frac{\sin^4(\pi f\mathrm{T})}{(\pi f\mathrm{T})^2}\,df \qquad (12)$$

Where $S_X(f)$ is the power spectral density of the random process $x(t)$. Eq. (12) states that, the Allan variance is proportional to the total power output of the random process when, passed through a filter with the transfer function of the form $\sin^4(x)/(x)^2$. Eq. (12) is the focal point of the Allan-variance method.

5. NOISE IDENTIFICATION USING ALLAN VARIANCE

In this section, a description of identification and estimation of various noise terms using a log–log plot of $\sigma(\mathrm{T})$ versus T is presented. Expressions for the PSD of different types of random processes are given and from them, the Allan Variance for the corresponding noise terms is calculated using Eq. (12). The following description will make it clear that different noise terms are represented by different slopes on a log–log plot of σ versus T. Thus, by noting down, the values of slopes present in that graph, the error terms present in the inertial sensor can be identified.

5.1 Angle (Velocity) Random Walk

The high frequency noise term that have correlation time much shorter than the sample time can contribute to the gyro angle (or accelerometer velocity) random walk. These noise terms are all characterized by a white-noise spectrum on the gyro (or accelerometer) rate output. The associated rate noise PSD is represented by:

$$S_y(w)=N^2 \qquad (13)$$

Where $S_y(w)$ is the PSD of white noise. Substituting Eq. (13) into equation (12), and performing the integration yields;

$$\sigma^2(\mathrm{T})=\frac{N^2}{\mathrm{T}} \qquad (14)$$

Then,

$$\sigma(\mathrm{T})=\frac{N}{\sqrt{\mathrm{T}}} \qquad (15)$$

As shown in Figure 5, Eq. (15) indicates that a log–log plot of $\sigma(\mathrm{T})$ versus T has a slope of $-1/2$. . Furthermore, the numerical value of N can be directly obtained by reading the slope line at T = 1.

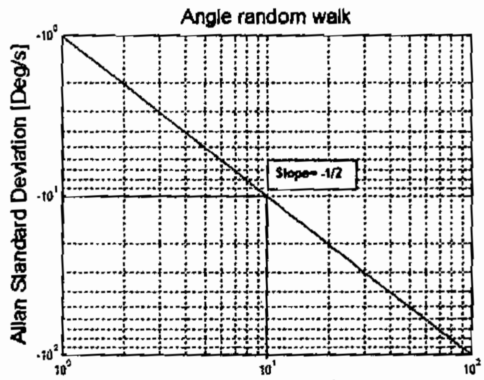

Fig.5 $\sigma(\mathrm{T})$ Plot for angle random walk (after IEEE Std.952-1997).

5.2 Rate Random Walk

The rate PSD associated with this noise is

$$S_y(w) = \frac{K^2}{w^2} \quad (16)$$

where K is the rate random-walk coefficient.

Substituting (16) into (12) and performing the integration yields;

$$\sigma^2_{Rw} = \frac{K^2 T}{3} \quad (17)$$

Then,

$$\sigma_{Rw} = K\sqrt{\frac{T}{3}} \quad (18)$$

This indicates that, the rate random walk is represented by a slope of +1/2 on a log–log plot of $\sigma(T)$ versus T, as shown in Figure 6. The magnitude of this noise, K, can be read off as the slope line at T = 3.

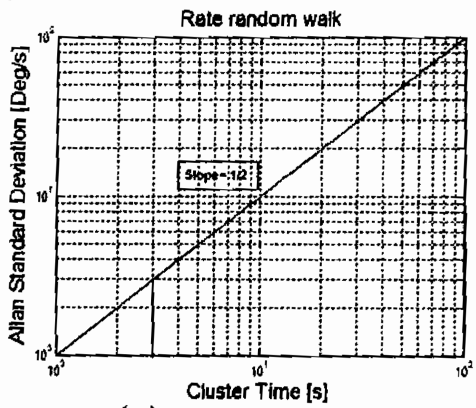

Fig. 6 $\sigma(T)$ Plot for rate random walk (after IEEE Std.952-1997).

5.3 Exponentially Correlated (Markov) Noise

This noise is characterized by an exponential decaying function with a finite correlation time. The PSD for such process is:

$$S_y(\omega) = \frac{2(T_c \sigma_{GMP})^2}{1+(\omega T_c)^2} \quad (19)$$

where σ_{GMP} is the noise amplitude and T_c is the correlation time.

Substitution of Eq. (19) in Eq. (12) and performing the integration yields:

$$\sigma^2(T) = \frac{(\sigma_{GMP} T_c)^2}{T}\left[1 - \frac{T_c}{2T}\left(3 - 4e^{-\frac{T}{T_c}} + e^{-\frac{2T}{T_c}}\right)\right] \quad (20)$$

Figure 7 shows a log-log plot of square root of Eq. (20). It is instructive to examine various limits of this equation. For T much longer than the correlation time, it is found that:

$$\sigma^2(T) \Rightarrow \frac{(\sigma_{GMP} T_c)^2}{T} \qquad for\ T \gg T_c \quad (21)$$

Which is Allan variance for angle (velocity) random walk where N = $\sigma_{GMP} T_c$ is the angle (velocity) random walk coefficient. For T much smaller than the correlation time, equation (20) reduces to:

$$\sigma^2(T) \Rightarrow \frac{\sigma_{GMP}^2}{3} T \qquad for\ T \ll T_c \quad (22)$$

which is the Allan variance for rate random walk. Table 1 gives the relationship of the various random processes and the Allan variance.

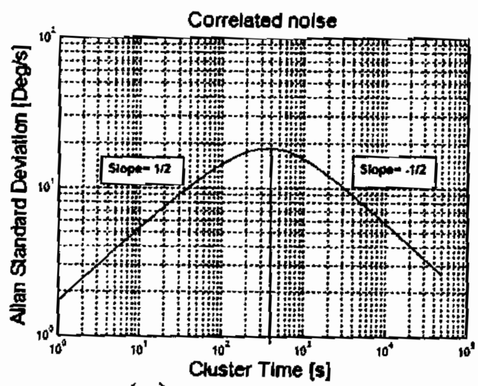

Fig.7 $\sigma(T)$ Plot for correlated noise (after IEEE Std.952-1997).

Noise type	Parameter of interest	Root Allan variance	Slop of root Allan variance plot
White noise	N	$\sigma_N = N/\sqrt{T}$	-1/2
Bias instability	B	$\sigma_B = B/0.6648$	0
Random walk	K	$\sigma_K = K\sqrt{T/3}$	+1/2
Correlated Noise	σ_{qc}, T_c	$\frac{(\sigma_{qc} T_c)^2}{T}, T \gg T_c$ $\frac{(\sigma_{qc})^2}{3} T, T \ll T_c$	∓1/2

Table 1 Summary of relation between Allan variance and different noise sources.

5.4 Combined Effects of All Processes

In general, any number of random processes discussed above (as well as others) can be present in the data. Thus, a typical Allan variance plot looks like the one shown in Figure 8. Experience shows that in most cases, different noise terms appear in different regions of (*T*). This allows easy identification of various random processes that exist in the data. If it can be assumed that, the existing random processes are all statistically independent; then, it can be shown that, the Allan variance at any given *T* is the sum of Allan variances due to the individual random processes at the same *T*. [4].

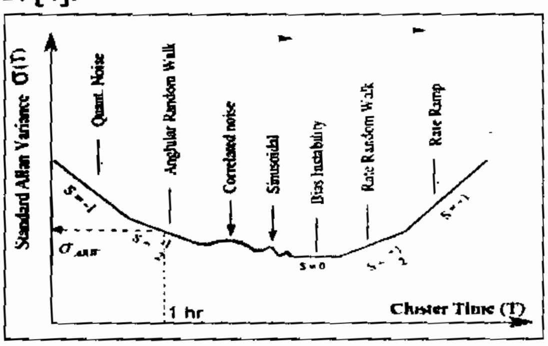

Fig. 8 σ (T) Allan variance analysis noise terms results (IEEE Std. 952-1995, 1998)

6. REAL DATA ANALYSIS

In this section, random noises of inertial measurement unit (IMU) type Xsen MTi will be analyzed using Allan variance method [7]. Xsen MTi IMU has three orthogonal angular rate gyros, three orthogonal accelerometers, three orthogonal magnetometers and a 16-bit A/D converter. Software MatLab program as a Graphical User Interface (GUI) was developed to calculate the Allan variance, autocorrelation function, and power spectral density of the collected data, and to simulate various inertial sensor errors for validation purpose. The equipment used in this test are shown in Figure 9.

Fig. 9 test set up to collect IMU data.

6.1 MTi Allan Vriance Analysis

Four hours static data were collected from the Xsen MTi IMU at 100Hz sampling rate. The raw data output from the IMU (voltage values) were first converted to acceleration and angular rate. By applying the Allan-variance method to the whole data set using the developed software Mat_Lab program, a log–log plot of the Allan standard deviation versus the cluster time is shown in Figure 10 for the gyro data. The results clearly indicate that the angel random walk noise is the dominant error for the short cluster times (slop -1/2), whereas the bias instability is the dominant error for the long cluster times (slop zero).

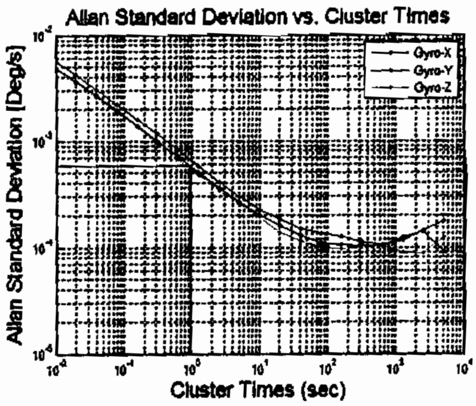

Fig. 10 Rate gyro Allan-variance results.

To extract the different noise parameters a straight line is fitted to the plot and the noise coefficient can be read out. For example, if the white noise coefficients should be obtained, a straight line with slop -1/2 would be fitted to the log-log plot of the square-root Allan variance. The white noise coefficient is obtained by reading the slope line at T=1 as shown in Figure 10. Therefore, the same procedure carried out in order to extract the noise parameters contaminated in the sensors data. The estimated noise coefficients for the rate gyro XYZ-axis are given in table 2.

Table 2 Identified error coefficients for Rate gyros XYZ-axis, using Allan variance.

Axis	Angel random walk $N[rad/\sqrt{s}]$	Bias instability $B[rad/s]$	Rate random walk $K[rad/s/\sqrt{s}]$
X	5.8e-4	1.1e-4	N/A
Y	6.6e-4	1.07e-4	N/A
Z	5.5e-4	9.047e-5	N/A

Similarly, the different errors of the accelerometers can be characterized through analyzing the curves in Figure 11. Table 3 lists all the identified error coefficients for the accelerometers.

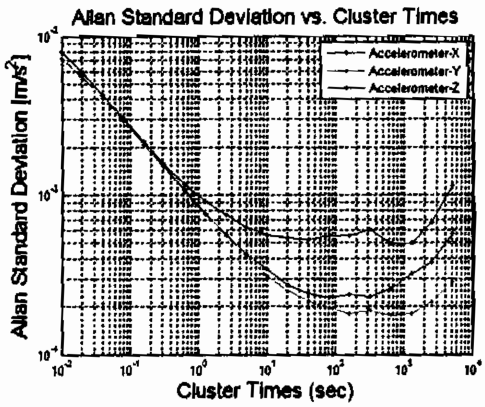

Fig. 11 Accelerometer Allan-variance results.

From Figure 11, one can find that, the white noise is dominant at short time clusters for the accelerometers, similar to the gyros. At long time clusters, the rate random walk noise is dominant for the accelerometers of XY-axis and the bias instability is almost dominant for the accelerometer of Z-axis.

Table 3 Identified error coefficients for accelerometers, using Allan variance.

Accelerometers	Velocity random walk $m/s^2/\sqrt{h}$	Bias instability m/s^2	Acceleration Random walk $m/s^2/s^{1/2}$
X	8.5e-4	2e-4	15e-6
Y	8.9e-4	1.7e-4	7.5e-6
Z	8.6e-4	5e-4	N/A

7. MODEL VALIDATION

The statistical characteristics achieved using Allan variance in fact represent the stochastic models parameters given in section (2). These stochastic models are used to simulate sensor stochastic errors for comparison to the experimental data from the actual sensors [5, 6]. Software Mat_Lab programs was developed to calculate the Allan variance, Autocorrelation function, power spectral density and to simulate various stochastic inertial sensor errors for validation purpose. Figure 12 shows the interface of these Mat_Lab programs as a Graphical User Interface (GUI).

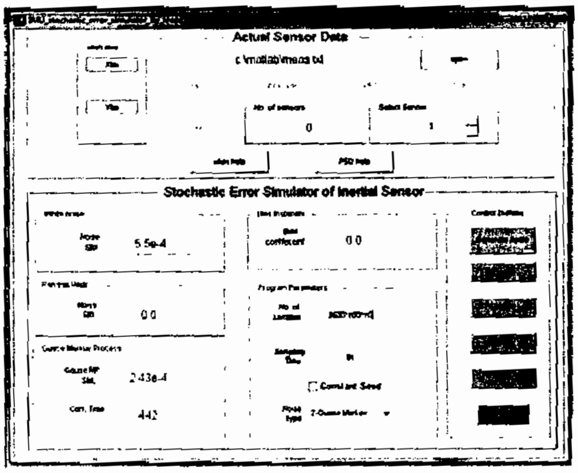

Fig. 12 stochastic error simulator for inertial sensors

The simulated sensor models, once verified, can be used by a navigation filter such as, Kalman filter, to provide optimized estimation of navigation parameters PVA.

For example, Using these Mat_Lab programs from the interface (GUI) shown in Figure 12. The actual sensor data of the Xsen IMU X-axial gyro was loaded from text file, and then the Allan variance of this experimental data was determined and plotted just by pressing the Key "Allan var" on the GUI; Figure 13 presents Allan variance of the experimental data (blue curve). The statistical parameters extracted from the determined Allan variance fed to the stochastic sensor simulator in order to generate simulated sensor data. The Allan variance of the simulated sensor output was plotted in the same Figure 13 for comparison purpose.

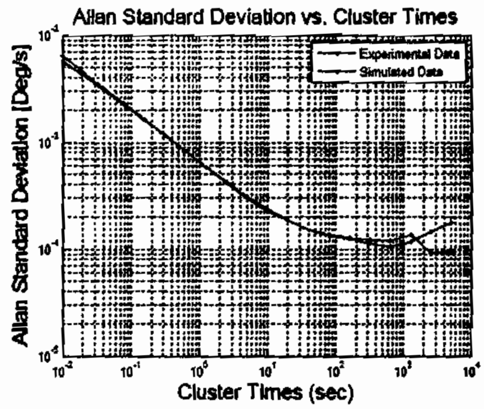

Fig. 13 Allan Variance Comparison of Experimental and Simulated Data

The close match of the Allan variance of the experimental data and the simulated data for gyro-X indicates that the estimated error coefficients for the Xsen MTi are correct, and sufficient to model its stochastic errors. The same process was carried out for all other sensors and the results were acceptable.

CONCLUSION

The Allan Variance is a simple and efficient method for identifying and characterizing different stochastic processes and their coefficients. Through some simple operations on the sensors output, a characteristic curve is obtained which used to determine the types and magnitudes of certain random errors possibly residing in the data. Therefore, the statistical parameters of the stochastic errors residing in the measurement data were determined using Allan variance.

Four hours static data have been collected from the Xsen MTi. The close match of the Allan variance of the experimental data and the simulated data indicates that the estimated error coefficients for the Xsen MTi are correct, and sufficient to model its stochastic errors. From the testing results the conclusion can be drawn that the dominant noise type is angel (velocity) random walk noise for both gyros and accelerometers for the short time clusters. For the long time clusters, the dominant error for the gyros is the bias instability but for accelerometers the dominant error is bias instability and rate random walk.

Finally, since the statistical parameters of the inertial stochastic errors identified and quantified, an error model can be derived, in order to be used as future work in the integrated navigation system.

REFERENCES

1. N. El-Sheimy, "Analysis and Modeling of Inertial Sensors Using AV", IEEE Transaction on instrumentation and measurement, vol. 57, No.1, JANUARY 2008.
2. Allan, D.W. "Statistics of atomic frequency standards " Proceedings of the IEEE, Vol. 54, No. 2, pp. 221–230, 1966.
3. J.W. Chaffee, "Relating the Allan variance to the diffusion coefficients of a linear stochastic differential equation model for precision oscillators," IEEE Trans. Ultrason., Ferroelectr., Freq. Control, vol. UFFC-34, no. 6, pp. 655–658, Nov. 1987.
4. IEEE Standard 952–1997 " IEEE Standard Specification Format Guide and Test Procedure for Single-Axis Interferometric Fiber Optic Gyros".
5. John H. Wall and David M. Bevly "Characterization of Inertial Sensor Measurements for Navigation Performance Analysis" GPS and Vehicle Dynamics Lab, Department of Mechanical Engineering, Auburn University, 2007.
6. Flenniken W. S. IV "Characterization of Various IMU Error Sources and the Effect on Navigation Performance " Proceedings of The Institute of Navigation's GNSS Meeting, Long Beach, CA, September 2005.
7. http:// www.xsens.com.
8. M. M. Tehrani, "Ring laser gyro data analysis with cluster sampling technique," *Proc. SPIE*, vol. 412, pp. 207–220, 1983.
9. Gelb, A., "Applied Optimal Estimation." The M.I.T. Press, Cambridge, Massachusetts, 1974.
10. Rogers, Robert M. "Applied mathematics in integrated navigation systems." 2nd ed. AIAA education series, 2003.
11. Brown R.G., Hwang P. "Introduction to Random Signals And Applied Kalman Filtering" Wiley. New York,1997.