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# ON THE PSEUDOMETRICS WITH HEINE-BOREL PROPERTY

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## ABSTRACT

The concept of HB-Pseudometric is introduced and investigated for arbitrary Tychonoff spaces. We prove that a space has HB-Pseudometric iff it is locally compact and σ-compact. Moreover, we study the WHB-Pseudometric and investigate some of their properties.

## INTRODUCTION

The concept of HB-metric is introduced in [2] and investigated in [4]. In this paper we shall introduce HB-Pseudometric on arbitrary Tychonoff space. A closed mopping f: X X⟶ Y is perfect if $f^{-1}(y)$ is compact for every $y \in Y$. A space which can be represented as a countable union of compact ( resp. countably compact) subspaces is called σ- compact ( resp. σ- countably compact). If the space X is locally compact and σ- compact then X can be represented as the union of an increasing sequence of open sets $U_n$ such that $cl_X U_n$ is compact and $cl_X U_n \subseteq U_{n+1}$ for every $n \in N$ (see [1] or [3] ). The symbol $cl_X U$ denotes the closure of a set U in X. A space X is called Pseudocompact if every continuous real - valued function defined on X is bounded. A space X is called locally countably compact if every point of X has a neighborhood U such that $cl_X U$ is countably compact [1]

## HB-PSEUDOMETRICS

A Pseudometric d on a topological space X is continuous if for every $x \in X$ and number $r > 0$ the set $S(x,r) = \{y \in Y : d(x,y) < r\}$ is open in X. a set $B \subset X$ is bounded relative to a Pseudometric d if there exist a point $x \in X$ and a number $r>0$ such that $B \subset S(x,r)$ and this is equivalent to sup $\{d()x,y) : x,y \in B \} < \infty$.

**Definition 1.1.** A Pseudometric d on X is said to be HB-Pseudometrics (or it satisfies the Heine- Borel property) if it satisfies the following conditions:

1. The Pseudometric d is continuous on X.
2. If the set $B \subset X$ is bonded by the Pseudometric d, then $cl_{XB}$ is compact.

**Remark 1.2.** Let d be a HB-Pseudometric on a space X. For every point $x \in X$ we put $H(x) = \{ y \in Y : d)x,y) = 0 \}$. Such a definition decompose the set X into compact subset$\{H(x):x \in X\}$. Let $X/d = \{H(x): x\in X\}$ and $\pi : X \longrightarrow X/d$ where $\pi^{-1}(\pi(x) = H(x)$. On X/d consider the following metric $d^*$ define as follows: $d^*(H(x), H(y)) = d(x,y)$ for every $x,y \in X$. then $(X/d, d^*)$ is a metric space and clearly is continuous mapping. We shall prove that $\pi$ is closed mapping. let $x_0 \in X$ and U be an open in X set such that $\pi^{-1}(\pi(x_0)) \subset U$. We shall prove that there exists $r > 0$ such that $S(x_0,r) \subset U$. Let us consider that $S(x_0,1/2n) \setminus U \neq \phi$ for all $n \in N$. Let $x_n \in \{x_n : n \in N\} \subset S(x_0,1)$ is closed, discrete and bounded. Since d is HB-Pseudometric, then the set $L = cl_X L$ is compact and thus is countably compact. Then $S(x_0, 1/2n) \setminus U = \phi$. Hence $S(x_0, 1/2n) \subset U$ for some $n \in N$. This shows that the mapping $\pi$ is closed ( see Theorem 1.4.13 in [1]). The following theorem is a fundamental result,

**Theorem 1.3.** If X is a Hausdoroff space, then the following statements are equivalent :

1. The space X is locally compact and $\sigma$-compact.
2. On X there exists HB-Pseudometric .

**Proof.** First we shall prove that 1. $\longrightarrow$ 2.

Suppose X is locally compact and $\sigma$- compact. If X is compact , then we can define $d(x,y) = 0$ for every x, $y \in X$. It is obvious that d is HB-Pseudometric on X. Suppose X is not compact. Then there exists a sequence $\{U_n : n \in N)$ of open in X sets such that $cl_X U_n$ is compact and $cl_X U_n \subset U_{n+1}$ for every $n \in N$. By the normality of X we construct continuous functions $f_n : X \longrightarrow [0,1]$, $n \in N$ such that $U_n \subset f_n^{-1}(0)$ and $X \setminus U_{n+1} \subset f_n^{-1}(1)$. put $d(x,y) = \{\sum \left| f_n(x) - f_n(y) \right| : n \in N \}$. If $x,y \in X$ , then there exists $n \in N$ such that $x,y \in U_n$. Then $f_i(x) = f_i(y) = 0$

for every $i \geq n$ and $d(x,y) = \sum \{ |f_i(x) - f_i(x_0)| : i \leq n \}$ defines a Pseudometric on X . we shall prove that d is HB- Pseudometric. Let $x_0$ be any point in X and $r>0$ be any number. Then there exists $n \in N$ such that $x_0 \in U_n$. The function $g_n(x)$ $0 \sum \{ |f_i(x) - f_i(x_0)| : i \leq n\}$ is continuous on X and $x_0 \in g_n^{-1}[0,r] \cap U_n \subset S(x_0,r)$. Thus $S(x_0, r)$ is open in X. Then d is continuous Pseudometric on X. Now let $a \in U_0$. Then $S(a,n) \subset U_n$. If the set H is bounded, then $H \subset S(a,n)$ for some $n \in N$ . Then $cl_X H \subset cl_X U_n$ and $cl_X H$ is compact. Hence d is HB-Pseudometric.

Conversely, we shall prove that $2. \longrightarrow 1.$

Suppose that d is HB-Pesudometric on X. Then the set $S(x,1)$ is open in X and the set $cl_X S(x,1)$ is compact. Then the space X is locally compact. Consider the space $(X/d,d^*)$ define in remark 1.2 and let $x_0 \in X$ be any point. Then for every point $x \in cl_X S(x_0,1)$ we have $\pi^{-1}(\pi(x)) = H(x) \subset cl_X S(x_0,1)$. This shows that the mapping $\pi$ is perfect. Since every metric space is paracompact, then X is paracompact also. Then $X = \cup \{W_i : i \in I\}$, where the sets $W_i$ are open, compact and $W_i \cap W_j = \phi$ for $i \neq j$ (see Th.5.1.27 in[1]). We shall prove that the set I is countable. Suppose that the set I is not countable. For every $i \in I$ let $x_i \in \pi^{-1}(W_i)$. Then $d(x_i,x_j) > 0$for every $i,j \in I$ and $i \neq j$. Let $i_0 \in I$ and $I_n = \{i \in I : n \geq d(x_{i_0},x_i) \geq 1/n\}$. It is clear that $I = \{i_0\} \cup \cup \{ I_n : n \in N\}$. Then there exists $n \in N$ such that the set $I_n$ is not countable. The set $H = \{x_i : i \in I_n \}$ is closed, bounded infinite and discrete. Then $cl_X H = H$ is not compact. This contradicts that d is HB- Pesudometric. Then I must be countable. This proves that the space X is $\sigma$-compact and the theorem is proved.

From the properties of perfect mappings we deduce that the space X/d is locally compact and $\sigma$-compact.

The following theorem is given in [4].

Theorem 1.4 A metric space (X,d) has a Heine-Borel metric which is locally identical to d if it is complete, $\sigma$-compact and locally compact.By theorem 1.3 and theorem 1.4 we have

**Corollary 1.5.** A Hausdoroff space X has a HB- Pesudometric iff there exists a perfect mapping from X onto a space with HB- metric.

**Proof.** Follows directly from theorem 1.3, 1.4 the theorem 3.7.21 and 3.7.24 given in [1].

**Theorem 1.6.** If $d_1$ is a continuous Pesudometric on X and $d_2$ is HB- Pesudometric on X then $d = d_1 + d_2$ is HB- Pesudometric. If $d_2$ is a metric, then d is metric.

**Proof.** If the set $L \subset X$ is bounded relative to the Pesudometric d then it is bounded relative to the Pesudometric $d_1$ and $d_2$.

**Definition 1.7.** We say that two Pseudometrics d and $d'$ on X are locally identifcal if every point $x \in X$ has a nieghbourhood $O_x$

**Example 2.1** Let X be a countably compact space. Then the function

$d(X,Y) = 0$ for every $X, Y \in X$

is WHB-pseudometric. If X is not compact, then d is not HB-pseudometric.

**Theorem 2.2** Let d be a continuous pseudometric on X. Then the following are equivalent

1- For every bounded set $L \subset X$, the set $cl_X L$ is countably compact.

2- For every bounded set $L \subset X$, the set $cl_X L$ is pseudocompact.

**Proof**: Since every Tychonoff countably compact space is pseudocompact (Theorem 3.10-20 in [3]), then $1. \rightarrow 2.$ we shall prove that $2. \rightarrow 1.$ Let $L \subset X$ be a bounded set and the set $H = cl_X L$ is not countably compact. Then in H there exists a countable discrete subset $E = \{X_n : n \in N\}$. The set E is a closed subspace of the space H. Then by Tietze-Urysohn Theorem there exists a continuous function $f:H \longrightarrow R$ such that $f(X_n) = n$ for $n = 1, 2, \ldots$ . But f is not bounce. Then H is not pseudocompact. The theorem proved.

A closed mapping $f: X \rightarrow Y$ is called quasiperfect if $f^{-1}(Y)$ is countably compact for every $Y \in Y$.

**Theorem 2.3** Let d be a WHB-pesudometric on the space X, then the following statements hold:

1- The mapping $\pi$ is quasi-perfect.

2- $d^*$ is HB-metric onto X/d.

3- The space X is locally countably compact

and on the pseudometrics. $\sigma$-countably compact.

**Proof:** Statement 1. is obvious from Remark 1.2 and that the set $\pi^{-1}(\pi(X))$ 0 { Y $\in$ X : d(X,Y) = 0} is countably compact. The statement 2. follows from 1. and the fact that every countably compact paracompact space is compact (see Theorem 5-1.20 in [3]).

Statement 3. follows directly from 1.

**Corollary 2.4.** A space X has a WHB-pseudometric if and only if there exists a quasi-perfect mapping onto a space with HB-metric.

**Theorem 2.5.** If $d_1$ is continuous pseudometric on X and $d_2$ is WHB-pseudometric on X, then $d_1 + d_2$ is WHB-pseudometric on X.

## COMPLETION OF PSEUDOMETRIC

In this section all spaces are considered to be Tychonoff unless stated otherwise. Let d be a continuous pseudometric on the space X. Consider a maximal set $X^d \supset X$ of the stone-Cech compactification $\beta X$ of the space X. The continuous pseudometric $\tilde{d}_{T}$s called the completion of the pseudometric d if d is the extension of d on $X^d$. If $X = X^d$, then the pseudometric d is called complete.

**Lemma: 3.1** The set $X^d$ exists and is unique.

**Proof:** Let $\pi : X \longrightarrow X/d = Y$ be the continuous projection from the space defined in Remark 1.2. By Hausdoroff's Theorem the metric $d^*$ on Y is extended to a complete metric $\tilde{d}$ on $\tilde{Y} \supset Y$, where Y is dense in $\tilde{Y}$. Let $\beta\pi : \beta X : \longrightarrow \beta\tilde{Y}$ be a continuous extension of $\pi$ , where $\beta X$ and $\beta\tilde{Y}$ are the Stone-Cech compactifications of the spaces X and $\tilde{Y}$ respectively. Let $X^d = (\beta\pi)^{-1}(\tilde{Y})$ and $d(x,y) = \tilde{d}(\beta\pi(x), \beta\pi(y))$ for every $x,y \in X^d$. The uniqueness and maximality of $\tilde{Y}$ implies that $X^d$ is unique and is maximal. A metric d on a space X is called K-metric if the projection $\pi : X \rightarrow X/d$ is perfect.

Since the projection $\tilde{\pi} : X^d \rightarrow X^d/\tilde{d}$ is perfect, the following theorem is true.

**Theorem 3.2** Every continuous pseudometric is extended to a complete K-metric.

**Theorem 3.3** The completion of WHB-pseudometric is HB-pseudometric and any HB-pseudometric is complete.

**Proof:** Let d be a WHB-pseudometric on the space X.Then by theorem 2.3 the mapping $\pi : X \rightarrow X/d$ is quasi-perfect and $\tilde{d}$ is complete HB-metric on $Y = X/d$. Hence $\tilde{\pi}$ maps perfectly $X^d$ onto Y, thus the mapping $\tilde{\pi} : X^d \rightarrow Y$ is perfect from $X^d$ onto Y. Consequently $\tilde{\pi} : X^d \rightarrow X^d/\tilde{d} = \tilde{Y}$ is a perfect mapping and $\tilde{d}$ is HB-pseudometric onto $\tilde{d}$.

To prove the second part of the theorem let d be a HB-pseudometric on X. Let $a \in X^d \setminus X$ be any point. Consider the sequence $L = \{ x_n : n \in N\} \subset X$ be such that $\tilde{d}(a,x_n) < 2^{-n}$. The set L is bounded and closed in X. Then L is compact subset. Since $\{ x \in X : \tilde{d}(a,x) = 0 \} \cap cl_{X^d} L \neq \phi$, then $cl_{X^d} L \neq L$. This implies that $X = X^d$. Then HB-pseudometrics are complete.

**Corollary 3.3** All HB-metrics are complete.

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