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Turnover of Stochastic Inventory Systems

إنسيابية نظم المخزون الإحتماليه

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خلاصة

إن تقييم نظم المخزون الفعلية باستخدام الطرق التقليدية يعد مهمة معقدة بسبب عدم ترابط متغيرات القرار وخاصة في نظم المخزون الشبكي. ولذلك يلجأ المتخصصون إلي استخدام معامل الانسيابية كأداة تقييم موضعية وكلية. وتبدو ندرة النماذج التحليلية التي تستخدم معامل الانسيابية وهذه الدراسة تقدم عدة نماذج علي أساس نظام المخزون المرجعي الاحتمالي بداية من دوال هدف متكاملة واعتمادا علي معامل الانسيابية وقد تم التركيز علي فلسفتين جديدتين لتبرير إسراع الانسيابية، أحداهما بنيت علي أساس التكلفة والأخرى علي أساس الربح.وهذه النماذج تشتمل علي بعض الصور المختصرة التي يمكن تطبيقها عمليا لتبسيط النظم الاحتمالية المعقدة. وقد تم تطبيق كل النماذج علي بيانات حقيقية وأخري فرضية في مجال واسع للمتغيرات. وقد أثبتت النتائج ربحية معامل الانسياب السريع وقد تم تفسير ذلك باستخدام منحنيات الربحية مع احتمالات عجز مختلفة في المدى المسموح به. وقد وجدت هذه المنحنيات منحرفة إلي يسار الربحية القصوى.وقد أظهرت المعادلات الإحصائية عمومية جميع متغيرات النظام مع اختلاف طفيف في الثوابت ويمكن إتباع نفس الإجراء لتصميم نماذج أخري لانسيابية النظم الاحتمالية أو الغير احتمالية.

Abstract

Evaluating performance of real inventory systems by using the conventional methods is a complex task because of disengagement of decision variables, especially for networks. Therefore, managers resort to the turnover rate (TOR) as a local and aggregate tool. Analytic models of TOR seem to be scarce in the literature. This study developed several TOR models, based on stochastic continuous-review system, starting from completely structured objective functions. Two novel philosophies are highlighted to justify rushing TOR from cost and profit views. The models include short ones that can be practically applied to abstract stochastic and sophisticated features of the system. The models are conducted to hypothetical and real data, followed by regression analyses. The results confirm that faster TOR is more profitable in a wide range. That is justified by the profitability curves with different allowable shortage probabilities and unit dynamic rates. Those curves are found skewed left to the optimum profitability. The explored statistical trends demonstrate generality for the system parameters that are merely modifying the trend coefficients. The same procedure can be followed to develop models for other deterministic and stochastic systems under several varieties of proceeding incidences.

Keywords: Inventory; Turnover rate; Stochastic demand; Rush; Return/Loss, Regression

1. Introduction

The conventional methods of inventory control measure the system performance through optimizing objective functions based on decision variables such as order quantity, maximum inventory level, cycle time, maximum shortage, and/or reorder level. However, a system is characterized by one or more of such variables, number of items, and status of demand (Elsayed and Boucher 1985; Soltan 1998). Almost, the objective is a cost function. The TOR is an idiom, among the vocabulary of inventory systems; it emulates the speed in mechanical systems. Generally, it is a function of the ratio, 'average annual demand to average inventory level,' in either dimension or dimensionless form. (See, e.g., Ballou 1981, Harhalakis et al. 1989, Wright 1992, Vergin 1998, Archibald et al. 1999, and Ballou 2000.) TOR is a powerful tool for simply auditing the system performance as a whole or at different stocking

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points of one or more items. Moreover, as will have seen later, TOR facilitates understanding and constructing the system components that needed to formulate a performance or objective function. This power comes from the fact that it explains several variables and parameters of the system. The system stays slow with small TOR and vice versa, searching for an optimum value. Thus, it becomes the language of practitioners. Evaluating real performance of a system with a tool but TOR becomes difficult; further, it becomes harder if the system is a network having multiple tiers (multi echelon and multi-item) as in spare parts disposition systems (Wright 1992; Ballou 2000).

Why TOR is crucial for inventory systems? Suppose that the system is a single-item single-echelon and basic EOQ fits ideally. In such situation, analysis based on order quantity confirms analysis based on TOR. But, in fact, when the system faces two opponents -human and nature, this assumption fails. In real situations, most of managers agree with necessity of analyzing TOR because of its drastic impact on the system profitability. From where change in profitability while return margin is fixed? This question has a unique answer that there are some hidden components of cost and profit depending on the system structure and incidences. Modeling a system as a function of TOR enables extracting explicit expressions for those components.

TOR analysis of inventory systems doesn't receive considerable attention in literature. Nevertheless, practitioners are usually anxious about that parameter. Ballou (1981) almost was the first effective report about TOR. He developed a generalized empirical formula to report the relationship between average inventory level and throughput—in aggregate and at different locations of inventory network. This formula is a compound linear-power function of average annual stock at each location. He held this relationship for available data on curves called '*turnover curves*' and followed regression analysis to compare his formula. Moreover, those TO-curves can be used to estimate the impact of system changes on performance and to set standard inventory levels. Ballou (2000) continued the work to demonstrate the practicality of TOR and TO-curves. Wright (1992) developed a nearly analytic formula for TOR, based on basic EOQ system, under cost and profit considerations. He used his formula to examine the disposition of spare parts between inventory echelons of a case assuming lognormal demand. Wright didn't derive his formula on basis of explicit stochastic variables.

This paper mainly follows analytic methodology, integrated with empirical routine, in order to solve for TOR as a decision variable. The focus is the commonly applied inventory system—stochastic continuous-review system (Q,r), where a fixed order Q is placed as inventory drops to reorder level r. As a result, lead time is always less than cycle time as shown in Fig. 1. It also assumes fixed order cost, fixed unit holding and shortage costs. Demand, lead time, and cycle time are stationary stochastic. Average inventory level depends on shortages' processing, i.e. backordered or not (lost sales). Basic structure of stochastic (Q,r) system can be explored from Elsayed and Boucher (1985), Winston (1994), and Soltan (1998). This system is analyzed in details in Zheng (1992), Zheng and de Groote (1993), and Axsäter (1996). It is also extended for more complexity such as Baker and Ehrhardt (1995), Chiang and Chiang (1996), Wang and Gerchak (1996), Agrawal and Seshadri (2000), Corbett (2001), and Johansen and Thorstenson (2004). This paper is a completely different direction since it reconstructs and analyzes the mentioned system using TOR beside other rational parameters.

The paper develops two categories of TOR models 'backordering' the shortages that may be incurred each cycle. First category, §2, includes two models based on different pragmatic philosophies that can be explained by natural incidences. Second category, §3, analyzes the complexity of first category and demonstrates other versions ranging from higher to lower complexities. The models are primarily developed to meet different demand distributions. A procedure is described for fitting to uniform and normal distributions. The same procedure can be followed to fit different demand distributions. However, an approximation is proposed to adjust skewed distributions relative to the normal distribution. The effect of main parameters on TOR are demonstrated, in §4, based on uniformally and normally distributed demands. Final discussion and concluding remarks are stated in §4. Appendix A contains complementary mathematical and statistical processes. All required nomenclatures and terms are set at the paper end.

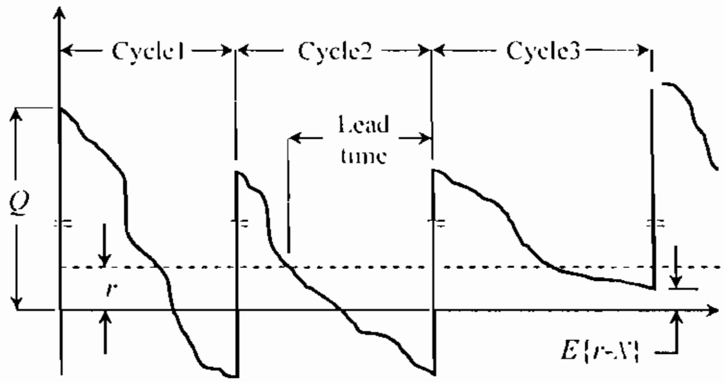

Fig.1. Stochastic continuous-review system (Q, r).

2. Basic TOR Models

Average inventory level of the described (Q, r) system is

$$\bar{I} = 0.5Q + r - E\{X\}.$$

Here, the dimensionless TOR form is adopted, formally

$$k = \frac{D}{0.5Q + r - E\{X\}}, \quad r \geq E\{X\}. \tag{1}$$

The condition $r \geq E\{X\}$ refers to the system stability. So, the range of k in formula (1) is tricky, since theoretically,

$$k \in [0, D/(r - E\{X\})] : Q \in [0, \infty].$$

Thus, $0 < k < D/(r - E\{X\})$ represents the real experimental range of a system. If the cyclic shortage isn't backordered, average inventory level modifies to

$$\bar{I} = 0.5Q + r - E\{X\} + E\{S\}.$$

Conventionally, the average annual inventory cost consists of three components (see Elsayed and Boucher 1984), ordering cost, holding cost, and shortage cost, formally

$$TC(Q,r) = \frac{D}{Q}(O + gE\{S\}) + wp(0.5Q + r - E\{X\}), \tag{2}$$

where the annual unit holding cost is often rated to the unit value; i.e., *wp*. *Substituting* from formula (1) and rearranging to

$$TC(k,r) = \frac{0.5D(O + gE\{S\})}{\left(\frac{D}{k} + E\{X\} - r\right)} + \frac{wpD}{k}. \tag{3}$$

Hence, the system becomes (k, r) instead of (Q, r).

2.1. Rush TOR models

This section presents two different pragmatic TOR models. Both models extend the typical model by adopting philosophies *rushing* the TOR (i.e., increasing its value). Management may have conservative views about nature of market and product beside change of storage environment. Intuitively, faster TOR leads to smaller cycles and larger number of orders. Therefore, save and/or gain of rushed TOR should be compared with associated costs of frequent reordering and probable shortage.

2.1.1. Rush of value-loss

The known holding cost refers to normal conditions of holding (see, e.g., Elsayed and Boucher 1985). The system future may be subjected to external random events, in addition to random demand, that leads to abnormal changes. Therefore, some of known sources of holding cost may behave randomly such as deterioration, technological depreciation, market competition, material cost, and economic instability. Thus, with slow TOR, original value of inventory may be reduced. That may drastically increases holding cost by a serious stochastic component. The risk of value-loss is assessed by the probability of one or more random events.

Proposition-1. Value-loss follows a random variable proportional to inverse of TOR and total value of inventory. Let χ be a random variable, $0 \le \chi \le 1$, representing unit *value-loss rate* with density function f_χ. Let ξ be a binary random variable, $\xi \in \{0, 1\}$, representing state of value-loss with mass function

$$f_\xi \in \{f_\xi(0), f_\xi(1)\}.$$

Let n be the maximum number of risky times in which the system may be subjected to value-loss. Then, $\sum \xi$ is a binomially distributed random variable, with parameters n and $f_\xi(1)$, representing the number of risky times. Let α be the expected value-loss rate per each slowing down, $0 \le \alpha$. Then,

$$\begin{aligned}\alpha &= \sum E\{\xi\chi\} \\ &= n f_\xi(1) E\{\chi\}\end{aligned},$$

which in turn, explains another binomial distribution with parameters n and $f_\xi(1)E\{\chi\}$. Under such conditions, the system is subjected to value-loss at different levels of inventory. Based on average inventory level, the average annual value-loss can be formulated as

$$\begin{aligned}C_a &= \alpha p(0.5Q + r - E\{X\}) \\ &= \alpha p \frac{D}{k}.\end{aligned} \qquad (4)$$

Since pD is the total value of inventory and α plays the role of proportion parameter, the proposition is proved. Even α may be a virtual parameter; it has a real effect on the system. In other words, it may be biased in cost estimation, but the error of assessing its value becomes insignificant in estimating k. As a result, the average annual inventory cost becomes

$$TC(k, r) = \frac{0.5D(O + gE\{S\})}{\left(\dfrac{D}{k} + E\{X\} - r\right)} + \frac{(w+\alpha)pD}{k}, \qquad (5)$$

Following differential calculus, optimal values of k and TC are obtained as

$$k_a^* = \frac{D}{\left(\dfrac{0.5D(O + gE\{S\})}{(w+\alpha)p}\right)^{0.5} + r - E\{X\}}, \qquad (6)$$

$$TC(k_a^*, r) = (w+\alpha)p\left(\frac{2D}{k_a^*} + E\{X\} - r\right). \qquad (7)$$

Also, the optimal value of r could be derived by applying differential calculus to formula (5). An obstacle exists that the optimal values of k and r constitute functions of each other; hence, there are no explicit expressions for both. Therefore, following recursive marginal analysis is necessary, which hardly complicates the solution and may lead to unacceptable values. (See marginal analysis in Winston 1994.) Referring to formula (2) and Appendix A1, there is trade-off between holding cost and shortage cost according to r—increasing r reduces shortage cost and increases holding cost, and vice versa. The author proposes that it is easier and applicable to follow a lower bound on r provided a specific demand distribution and an upper bound on the probability of shortage. Also, k generates a trade-off between holding and shortage cost but with reversed effect. Concerning cost trade-off, such rush should be compared with incurred shortage cost and its effect on the total cost function. The term $(w+\alpha)p$ plays a role as unit holding cost—deterministic component plus stochastic component. Intuitively, $(w+\alpha) = 1$ consumes the unit value in the average inventory level that becomes a burden on the system. The case becomes worst when $(w+\alpha) > 1$, thus necessitates setting faster k. to confront malicious future incidences.

2.1.2. Rush of dynamic value-return

This section models TOR from the profitability aspect. It is known by managers' experience that faster TOR is more profitable (Wright 1992). Thus, maintaining inventory for only short cycles is preferred if the gain can recover ordering and shortage costs. As long as the selling price is almost constant, such increase of profit explains a hidden component of return. (This component is thought hidden because there are no similar explicit parameters or mathematical expressions in the literature.) The author calls this component '*dynamic value-return*,' and calls associated system '*dynamic return inventory system*.'

Proposition-2. Dynamic value-return follows a random variable proportional to TOR and total value of inventory. Let β be the expected dynamic value-return rate per each speeding up, $0 \le \beta \le 1$. The average annual dynamic value-return can be formulated as

$$P_\beta = \beta p D k \, . \tag{8}$$

The parameter β also has some virtual properties. Notice that average inventory level isn't the principal of proposing P_β in spite of its inclusion in k. Further, the concepts of α and β are exclusively adopted to rush TOR. The annual normal value-return is

$$P_\pi = \pi p D \, , \tag{9}$$

where π is the normal unit value-return rate. Then, the average annual profit yields

$$\begin{aligned} Z(k,r) &= (\pi p D + \beta p D k) - \frac{0.5D(O + gE\{S\})}{\left(\dfrac{D}{k} + E\{X\} - r\right)} - \frac{wpD}{k} \\ &= pD\left(\pi + \beta k - \frac{w}{k}\right) - \frac{0.5D(O + gE\{S\})}{\left(\dfrac{D}{k} + E\{X\} - r\right)} \cdot \end{aligned} \tag{10}$$

Appendix A2 shows the difficulty of getting an exact optimal value of k, while derives a lower limit such that

$$k_\beta^{*lower} = \frac{D - \left(\dfrac{0.5D(O + gE\{S\})}{\beta p}\right)^{0.5}}{r - E\{X\}} \, , \tag{11}$$

which can be used to find, graphically, the exact value—at maximum value of Z. Hence, the optimal value of average annual profit becomes exactly

$$\begin{aligned} Z(k_\beta^*, r) = pD\left(\pi + \beta k_\beta^* - \frac{w}{k_\beta^*}\right) - \\ \frac{p(w + \beta k_\beta^{*2})}{\left(\dfrac{D}{k_\beta^*} + E\{X\} - r\right)} \end{aligned} \cdot \tag{12}$$

2.2. Fitting to demand distributions

The models of §2.1 were developed in general distribution forms. In other words, the probability distributions of demand during lead time weren't specified. This section follows a procedure to fit those models to uniform and normal distributions, based on maximum probability of shortage per cycle. This probability represents a '***service level measure***' (SLM) (Zipkin 1986). Such concept is adopted, in this paper, as a handmaiden to approximate ***r*** and $E\{S\}$. This procedure also suits other demand distributions, may be with extensive statistics. Therefore, the author proposed a transformation for unimodal distributions, which enables switching from normal demand assumption with minimum statistical work. All inequalities based on SLM are set at equality sense, which equivalently converts the analysis of r to an analysis of ε. Thus making it possible to substitute smoothly in the corresponding formulas. Such SLM sets $E\{S\} = \Delta_1\sigma_X, \Delta_1 \ge 0$ and $r - E\{X\} = \Delta_2\sigma_X, \Delta_2 \ge 0$, for all demand distributions, where Δ_1 and Δ_2 are multipliers. (See Appendix A3 for instance.) Consequently, expected annual shortage could be expressed similarly. Generally, 'for all demand distributions, expected shortage per cycle under service measure can be a multiplier by the standard deviation of lead time demand.' Later, in §4, σ_X is reported as a function of D.

2.2.1. Uniform distribution

Suppose that demand during lead time is uniformally distributed random variable. Appendix A3.1 estimates $E\{X\}$, $E\{S\}$, and a lower limit of r providing an SLM of ε. If the value of r is set at equality sense, thus

$$k_\alpha^* = \frac{D}{\left(\frac{0.5D[O+0.5g\varepsilon^2(b-a)]}{(w+\alpha)p}\right)^{0.5} + (b-a)(0.5-\varepsilon)}, \tag{13}$$

$$TC(k_\alpha^*,\varepsilon) = (w+\alpha)p\left(\frac{2D}{k_\alpha^*} + (b-a)(\varepsilon-0.5)\right), \tag{14}$$

$$k_\beta^{*lower} = \frac{\left(\frac{0.5D[O+0.5g\varepsilon^2(b-a)]}{\beta p}\right)^{0.5} - D}{(b-a)(\varepsilon-0.5)}, \tag{15}$$

$$Z(k_\beta^*,\varepsilon) = pD\left(\pi + \beta k_\beta^* - \frac{w}{k_\beta^*}\right) - p(w+\beta k_\beta^{*2})\left(\frac{D}{k_\beta^*} + (b-a)(\varepsilon-0.5)\right). \tag{16}$$

2.2.2. Normal and other distributions

Suppose that demand during lead time is normally distributed random variable. Appendix A3.2 estimates a lower limit of r and an upper limit of $E\{S\}$ providing an SLM. If both values are set at equality sense, thus

$$k_\alpha^* = \frac{D}{\left(\frac{0.5D[O+g\sigma_X(0.4e^{-0.5\lambda_\varepsilon^2} - \varepsilon\lambda_\varepsilon)]}{(w+\alpha)p}\right)^{0.5} + \lambda_\varepsilon\sigma_X}, \tag{17}$$

$$TC(k_\alpha^*,\varepsilon) = (w+\alpha)p\left(\frac{2D}{k_\alpha^*} - \lambda_\varepsilon\sigma_X\right), \tag{18}$$

$$k_\beta^{*lower} = \frac{D - \left(\frac{0.5D[O+g\sigma_X(0.4e^{-0.5\lambda_\varepsilon^2} - \varepsilon\lambda_\varepsilon)]}{\beta p}\right)^{0.5}}{\lambda_\varepsilon\sigma_X}, \tag{19}$$

$$Z(k_\beta^*,\varepsilon) = pD\left(\pi + \beta k_\beta^* - \frac{w}{k_\beta^*}\right) - p(w+\beta k_\beta^{*2})\left(\frac{D}{k_\beta^*} - \lambda_\varepsilon\sigma_X\right). \tag{20}$$

Furthermore, similar models can be developed for other symmetric or asymmetric demand distributions by repeating the same fitting procedure. Appendix A3.3 proposes an approximation for nonlinear asymmetrical distributions relative to normal distribution. The principal is to deal with the distribution as if it is normal, and then a correction factor, δ, is applied. Value of δ measures the biasness from normal distribution. Nonlinear symmetrical distributions can be dealt as normal with acceptable errors. Generally, the original models can be directly applied to any distribution especially those discrete and multimodal by calculating $E\{X\}$, σ_X, and $E\{S\}$. However, fitting to a distribution is advantageous for studying the effect of distribution parameters on TOR.

3. Extreme TOR Models

This section discusses the TOR based on analyzing the reorder level r between two extremes, completely analytical models and simple models. TOR models developed in §2 were fitted to demand distributions based on SLM to estimate r and $E\{S\}$. If the optimal value of r is determined following a complete analytical routine, then $E\{S\}$ is obtained as a result. Appendix A4 exhibits using calculus for this purpose. The optimal values of r, which couple the models defined by formulas (6), and (11), are respectively

$$r_\alpha^* = \frac{D}{k_\alpha^*} + E\{X\} - \frac{0.5gD}{(w+\alpha)p}\int_{r_\alpha^*}^{\infty} f(x)dx, \tag{21}$$

$$r_\beta^* = \frac{D}{k_\beta^*} + E\{X\} - \frac{0.5gD}{(w+\beta k_\beta^{*2})p} \int_{r}^{\infty} f(x)dx. \quad (22)$$

Now we have different two equation systems: [a] (A1-2), (21) and (6); and [b] (A1-2), (22) and (11). Each equation system can be iteratively solved for optimal r and k, starting with $E\{S\}=0$ until reaching convergence. Generally, Eqs (21) and (22) can be fit to any demand distribution. There are two complexities inherent in using this procedure—iterative process and integral part. The integral part is a function of r, which complicates solving for r.

Work can be simplified by adopting SLM of ε for shortage. Since the integral part represents the probability of shortage in each cycle, it can be replaced with ε. That adds some simplicity to solve for r. Furthermore, to increase simplicity, the iterative procedure is dispensed to obtain '*crude values*' for r and k by replacing r, in equality sense, in formulas (21) and (22), according to the demand distribution used (Appendix A3). Hence, we can solve easily for k. If formulas (21) and (22) are fit to uniformally distributed demand with rearrangement, we get

$$\hat{k}_\alpha = \frac{D}{(b-a)(0.5-\varepsilon) + \dfrac{0.5\varepsilon gD}{(w+\alpha)p}}, \quad (23)$$

$$\hat{k}_\beta = \frac{D}{(b-a)(0.5-\varepsilon)}\left(1 - \frac{0.5\varepsilon g}{p(\beta\hat{k}_\beta + w/\hat{k}_\beta)}\right). \quad (24)$$

formula (23) directly gives crude values for k while formula (24) seems difficult to solve directly. However, using differentiation, the term $\beta k + w/k$ reaches its minimum value at $k=(w/\beta)^{0.5}$; making this substitution only in the right hand side yields a lower limit as

$$\hat{k}_\beta^{lower} = \frac{D}{(0.5-\varepsilon)(b-a)}\left(1 - \frac{0.25\varepsilon g}{p\sqrt{w\beta}}\right). \quad (25)$$

Following the same procedure for normally distributed demand, we get

$$\hat{k}_\alpha = \frac{D}{\lambda_\varepsilon \sigma_X + \dfrac{0.5\varepsilon gD}{(w+\alpha)p}}, \quad (26)$$

$$\hat{k}_\beta = \frac{D}{\lambda_\varepsilon \sigma_X}\left(1 - \frac{0.5\varepsilon g}{p(\beta\hat{k}_\beta + w/\hat{k}_\beta)}\right). \quad (27)$$

Similarly, substitute $k=(w/\beta)^{0.5}$, then

$$\hat{k}_\beta^{lower} = \frac{D}{\lambda_\varepsilon \sigma_X}\left(1 - \frac{0.25\varepsilon g}{p\sqrt{w\beta}}\right). \quad (28)$$

Similar work can be done for other demand distributions to get crude values for both r and k. However, to relate asymmetric distributions to normal distribution, managers could accept the factor δ. The multiplier $\delta\lambda_\varepsilon$ replaces λ_ε in the corresponding equations in this section (see Appendix A3.3). The simple models of TOR developed in this section may attract practitioners although they aren't completely analytical.

4. Application and Regression Analyses

Fitting the developed TOR models to specific distributions comprises the standard deviation of demand during lead time as a parameter in each model. Therefore, it is necessary to specify the relationship between this standard deviation and average annual demand.

Proposition-3. It is empirically reported that standard deviation of annual demand is a power function of its mean (Herron 1976). Consequently, since standard deviation of demand during lead time depends on the average annual demand, it can be approximated as $\sigma_X = \omega D^\gamma$. Where $0 \le \omega$ and $0 \le \gamma \le 1$ are constants. This relationship is found amenable to a large variety of unimodal distributions with changes in ω and γ. Thus, values of ω and γ explain the type of density function of demand. The value of ω can be fixed for all distributions adjusting the value of γ. This power relationship is yielded statistically aided by available huge data. The author doesn't confirm this proposition to the discrete distributions.

The primal objective is to examine the effect of average annual demand on TOR models in an environment of α, β, and ε. Intuitively, study of ε is equivalent to study of r; review Appendix A3. Statistical analysis is primarily conducted aided by proposition-3—assuming uniformally and normally distributed lead time demands—to the models by formulas (13), (15), (17), (19), (23), (24), (26), and (28). Other theoretical distributions such as lognormal are experimented using the original models in addition to corrected fittings (relative to normal distribution by value of δ). The parameters of each model are substituted over wide ranges of hypothetical and field values. Hence, regression analyses are carried out. All regression analyses yielded the same strong relationship ($R^2 \cong 1$) for all models and all distributions such that $k^*_{\alpha;\beta}; \hat{k}_{\alpha;\beta} = uD^v$, as shown in Fig. 2. Where, u and v are constant for a specific distribution over a range of D given values of other parameters. Moreover, u and v are found strong *parabolic functions* ($R^2 \cong 1$) of ε. Different plots are made for optimal and crude TOR's versus D, which found versions of the pattern of Fig. 2. Ranges of TOR's that registered in Fig. 2 increase with augmented α or β. Fig. 3 indicates that β registers optimal TOR's higher than α while change of α is stronger than change of β. Change of both parameters becomes stronger with higher values of ε. At each value of ε, values of α greater than 1 have significant effect on TOR while the effect of β asymptotes at 1. Crude TOR's behave similar to optimal TOR's except they register higher ranges. For uniform distribution, all models register smallest ranges of TOR's.

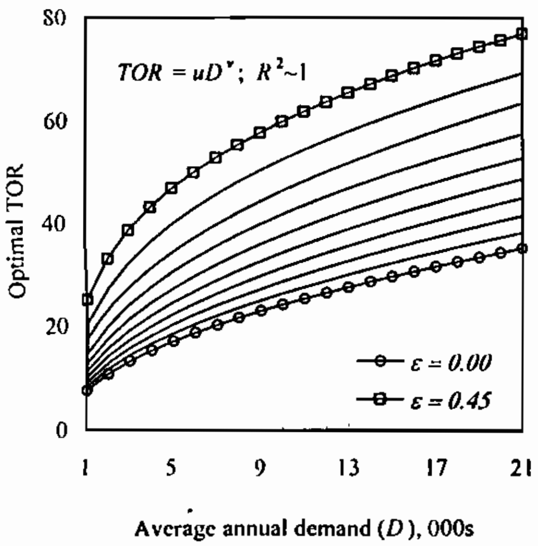

Fig. 2. TOR models with ε's folds given α or β.

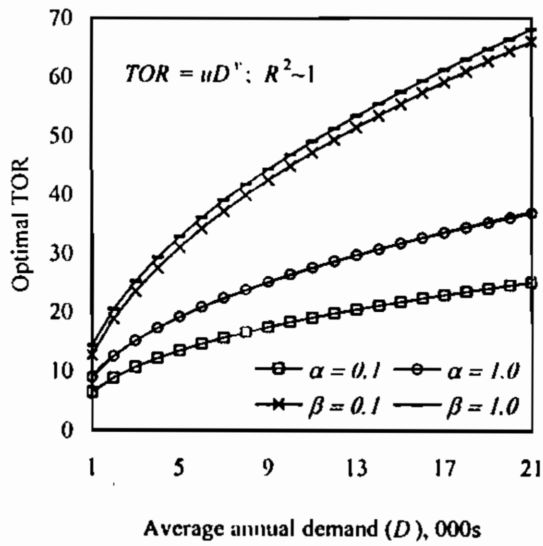

Fig. 3. TOR models with α's and β's folds given ε.

Furthermore and connected to the primal objective, it is necessary to examine the effect of TOR on the *profitability* as a measure of system performance. Some economical measures of performance were discussed in Arcelus and Srinivasan (1987). Let η be the system profitability such that $\eta = Z/(TC + pD)$. Next plots are also samples of plots made with wide ranges of system parameters. The experiment is carried out aided by the profit TOR model defined by formulas (10) and (11). Fig. 4 analytically proves believes of practitioners, that faster TOR is more profitable (Wright 1992). At each value of ε, increase of TOR yields increase of η up to optimum point (k^*_β, η^*) after which η drops drastically in shorter range. Also, this occurs at each value of β (Fig. 5). The stream of $((k^*_\beta, \eta^*), \varepsilon)$ fits a second order polynomial approaches a straight line as shown in Fig. 4. The stream of $((k^*_\beta, \eta^*), \beta)$ fits a slightly positive straight line as shown in 5. Both streams are dismembered in Figs. 6 and 8 respectively. Profitability curves in Figs. 4 and 5 fit strong high order polynomials. Moreover, those curves show that η is a negatively skewed polynomial of k_β with high order (about sixth). At each value of ε, change of profitability with TOR explains trade-offs between holding cost in a panel and order cost, shortage cost, and dynamic return in another panel. Sum of holding cost and shortage cost becomes dominant after the optimum point. The curves in Figs. 2 and 3 can be said as optimal TO-curves.

Fig. 6 dismembers the stream of $((k^*_\beta,\eta^*),\varepsilon)$, in Fig.4, into polynomials of ε with high order (about sixth). Both trends are strong and positively correlated. From the view of profitability, Fig. 7 demonstrates that difference between maximum and optimum TOR's isn't so high up to $\varepsilon = 0.20$ while reaches its maximum value, about 35%, around $\varepsilon = 0.50$. Their trends follow high order polynomials of ε and their ratio is approximately parabolic of ε. That supports the scenario appeared in Figs. 4 and 5. As cited before, maximum TOR is determined by ε (allowed shortage probability of cycles) and the parameters of stochastic demand during lead time. Also, the stream of $((k^*_\beta,\eta^*),\beta)$, in Fig. 5, can be dismembered and discussed through Fig. 8. Notice that change of $\beta\backslash\varepsilon$ doesn't significantly affect k^*_β although this affects η^* within a positive straight line. The skewness of profitability curves shown in Figs. 4 and 5 can be more explored from Fig. 9. The skewness of η as a function of β behaves smoothly with steepest ascent just after zero β. Logic of dynamic return appears in the steepest increase of skewness by just having β after zero. If zero β is excluded, skewness of η would follow a strong third order polynomial of β. Also, if ε's less than 0.15 are excluded, skewness of η would follow a strong second order polynomial of ε. Skewness of η as function of ε fluctuates up to ε about 0.15 and continues smooth after.

Changes of other parameters—order cost, unit value, unit shortage cost, and distribution parameters—don't violate the reported statistical trends. Significant changes of system parameters just modify the coefficients of a trend and keep strong correlation. For instance, the power relationship between TOR and D only modifies the values of u and v. In other words, all parameters scale the trends while maintaining similar behaviors. Analytically, stability of the system follows that $r \geq E\{X\}$ which is kept unrestricted throughout this paper. Nevertheless, adopting high SLM of ε may drive the system to work at unexpected shortage conditions especially around $\varepsilon = 0.5$. The system demonstrates real stability by setting ε below 0.5 while 0.2 or below secured the best stability in all models for all ranges of other parameters (Fig. 9).

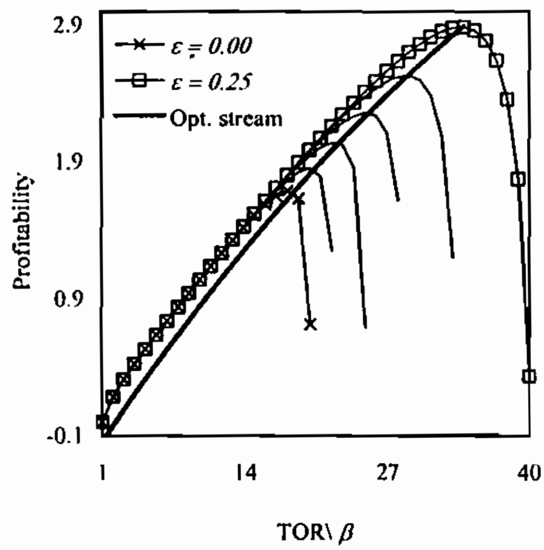

Fig. 4. Profitability with ε's folds given β.

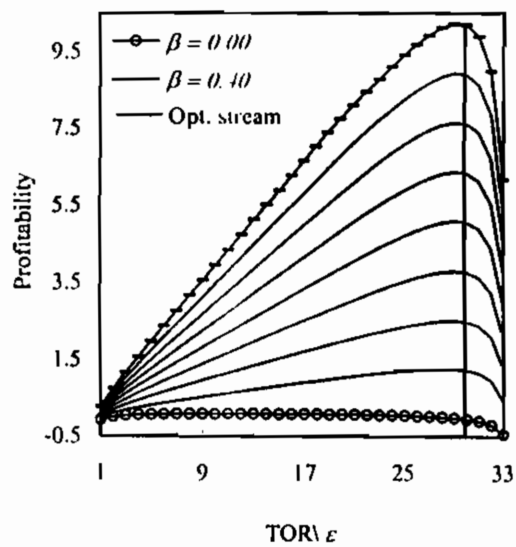

Fig. 5. Profitability with β's folds given ε.

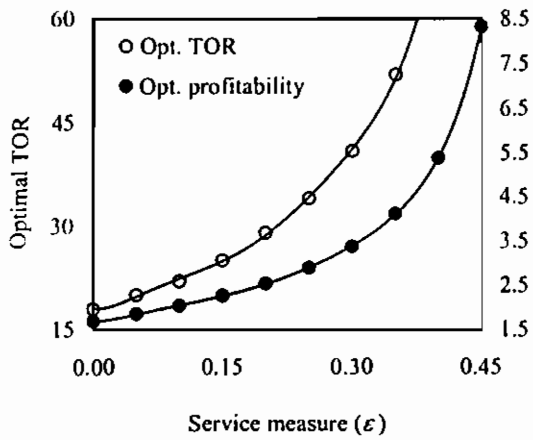

Fig. 6. Optima of TOR and profitability vs. ε.

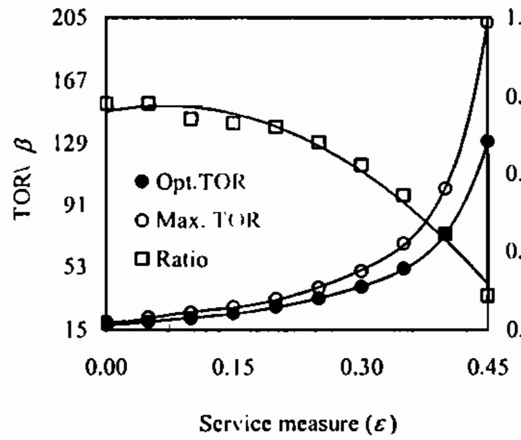

Fig. 7. Max. and opt. TOR's vs ε.

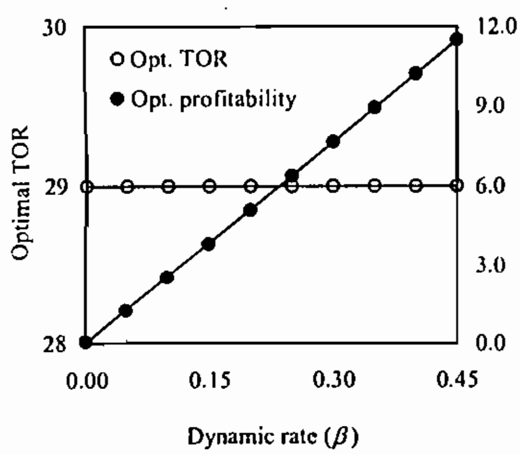

Fig. 8. Optima of TOR and profitability vs. β.

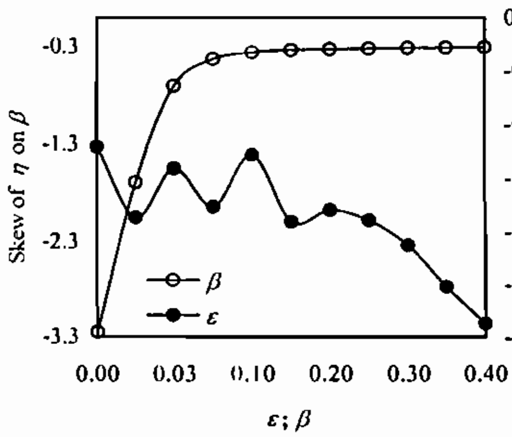

Fig. 9. Skewness of profitability curves.

5. Conclusions

Inventory systems realize review and ordering discipline of stock control. A system comprises one or more featuring parameters such as order quantity, review level, review period, and maximum inventory level. Those parameters don't demonstrate the system dynamics. TOR combines other featuring parameters in addition to some parameters of demand distribution. Therefore, TOR can be used as a comprehensive featuring parameter instead of classic parameters. It has the property of locality and aggregation for multi-echelon systems. The main objective has been to develop analytical models switching to TOR analyses of stochastic continuous-review system. That facilitates the study of system dynamics and performance. Two categories of models have been developed. Total inventory cost and profitability are adopted as measures of performance reflecting the TOR impact. The study shows how to fit those models to different demand distributions guided by the uniform and normal distributions provided a service level measure of shortage. Thus, two critical terms can be expressed as explieit functions of standard deviation of lead time demand—expected shortage per cycle and review level minus expected lead time demand. Expected shortage per cycle can be approximated by rating the standard deviation of lead time demand. For aggregate and multiple stock positions, experimental TO-curves can be constructed from available data about average annual demand and average inventory level. If a company doesn't apply a suitable inventory system, those curves may follow different trends with inconvincible correlation. Therefore, analytical curves are necessary as standards. Analytical TO-curves simply fit formula (1) with frequent modification. Here, two classes of analytical curves—optimal TO-curves and profitability curves—have been developed. Review Figs. 2 to 5. Practitioners deem faster TOR as more profitable than slower TOR. This belief has been analytically confirmed, up to a specific point, as shown by the developed profitability curves. However, change of profitability as TOR changed is explained by inherent trade-offs between different cost components. The regression analyses dedicate several trends for the essential system parameters. Such trends are found stable, since they are changeable only in coefficients over wide ranges of parameters. Salient is that optimum TOR fits a power function of the average annual demand. This paper possibly answers several questions about produced TOR, ideal turnover curves (linear or nonlinear) and their coefficients, inventory investment impact with TOR change, and currently applied inventory system (see Ballou 2000). Furthermore and important upshot, TOR is recommended for analytical or empirical outlining the system components and performance. Thus, the developed procedures and proposed tools are handmaidens for other deterministic and stochastic systems. The paper places a comprehensive methodology for analyzing and adjusting a system coinciding with the nature of items, storage facilities, and markets.

Appendix A

A1. Shortage estimation

For a review level, $r \geq 0$, and a random lead time demand, $X \geq 0$, the shortage and expected shortage per cycle are respectively

$$S(x) = \begin{cases} 0, & x \leq r \\ x-r, & x > r \end{cases}, \tag{A1-1}$$

$$E\{S\} = \int_0^{\infty} S(x)f(x)dx = \int_r^{\infty}(x-r)f(x)dx = \int_r^{\infty} xf(x)dx - r\int_r^{\infty} f(x)dx$$

$$= \int_r^{\infty} xf(x)dx - rP\{X \geq r\}. \tag{A1-2}$$

A2. Optimal TOR with dynamic value-return

First and second derivatives, respect to k, of the profit function described by formula (10) are

$$\frac{\partial Z(k,r)}{\partial k} = pD\left(\frac{w}{k^2}+\beta\right) - \frac{D^2(O+gE\{S\})}{2k^2\left(\frac{D}{k}+E\{X\}-r\right)^2}, \tag{A2-1}$$

$$\frac{\partial^2 Z(k,r)}{\partial k^2} = -2pD\left(\frac{w}{k^3}\right) + \frac{D^2(O+gE\{S\})(E\{X\}-r)}{k^3\left(\frac{D}{k}+E\{X\}-r\right)^3}. \tag{A2-2}$$

Since Z is maximized, and the second derivative is always negative (i.e., Z is a concave function), the optimal value of k satisfies

$$0 = p(w+\beta k^2) - \frac{D(O+gE\{S\})}{2\left(\frac{D}{k}+E\{X\}-r\right)^2}, \tag{A2-3}$$

that is difficult to solve for k. A lower limit for optimal k can be got at $w = 0$, thus

$$0 = \beta pk^2 - \frac{D(O + gE\{S\})}{2\left(\frac{D}{k} + E\{X\} - r\right)^2}, \tag{A2-4}$$

which is rearranged to yield formula (11). Since w is a deterministic fraction, it can be neglected in the term $\beta k^2 + w$. That if w is actually very small, formula (11) would be exact optimal value.

A3. Fitting to demand distributions

A3.1. Uniform demand

If demand during lead time, X, is uniformally distributed random variable, then mean and standard deviation of demand, and expected shortage per cycle are respectively

$$E\{X\} = 0.5(b + a);\ \sigma_X = 0.29(b - a),\ b \ge a \ge 0; \tag{A3-1}$$

$$E\{S\} = \int_r^b (x - r) f(x) dx = \int_r^b \frac{x - r}{b - a} dx = 0.5 \frac{(b - r)^2}{(b - a)},\ b \ge r \ge a \ge 0; \tag{A3-2}$$

which shows that $E\{S\}$ is a function of r. Notice that $E\{S\} \le E\{X\}$ from the ratio

$$\frac{E\{S\}}{E\{X\}} = \left\{\frac{b - r}{b - a} \times \frac{b - r}{b + a}\right\} \le 1,\ b \ge r \ge a \ge 0, \tag{A3-3}$$

Suppose that ε is an upper limit on the accepted probability of shortage during lead time, in each cycle, then

$$P\{X \ge r\} \le \varepsilon \to \left\{\frac{1}{b - a}\int_r^b dx\right\} \le \varepsilon \to \frac{b - r}{b - a} \le \varepsilon \to r \ge b - \varepsilon(b - a),\ b \ge a \ge 0,\ 0 \le \varepsilon \le 1. \tag{A3-4}$$

By equality sense of formula (A3-4), then as an upper limit

$$E\{S\} = 0.5\varepsilon^2(b - a) = 1.73\varepsilon^2\sigma_X. \tag{A3-5}$$

A parameter such as ε, or sensibly $1 - \varepsilon$, is known, in the context of inventory, as '***service level measure***' (SLM). If $r \ge b$, there is no chance for shortage occurrence (i.e., $\varepsilon = 0$).

A3.2. Normal demand

Following SLM of ε for normally distributed demand leads to

$$P\{X \ge r\} \le \varepsilon \to P\left\{\frac{X - \mu_X}{\sigma_X} \ge \frac{r - \mu_X}{\sigma_X}\right\} \le \varepsilon \to \frac{r - \mu_X}{\sigma_X} \ge \lambda_\varepsilon \to r \ge \mu_X + \lambda_\varepsilon \sigma_X,\ -3 \le \lambda_\varepsilon \le 3. \tag{A3-6}$$

Where, λ_ε is the standard normal multiplier having ε area to right. Expected shortage per cycle is given by

$$\begin{aligned} E\{S\} &= \frac{0.4}{\sigma_X}\int_r^\infty (x - r) e^{-0.5[(x-\mu_X)/\sigma_X]^2} dx \\ &= \frac{0.4}{\sigma_X}\int_r^\infty x e^{-0.5[(x-\mu_X)/\sigma_X]^2} dx - \frac{0.4r}{\sigma_X}\int_r^\infty e^{-0.5[(x-\mu_X)/\sigma_X]^2} dx \\ &= \frac{0.4}{\sigma_X}\int_r^\infty x e^{-0.5[(x-\mu_X)/\sigma_X]^2} dx - rP\{X \ge r\}. \end{aligned}$$

Using the substitutions $y = (x - \mu_X)/\sigma_X$, $y: y_0 \to \infty$, and $y_0 = (r - \mu_X)/\sigma_X$, then

$$\begin{aligned} E\{S\} &= 0.4\int_{y_0}^\infty (y\sigma_X + \mu_X) e^{-0.5y^2} dy - rP\{X \ge r\} \\ &= 0.4\sigma_X \int_{y_0}^\infty y e^{-0.5y^2} dy + 0.4\mu_X \int_{y_0}^\infty e^{-0.5y^2} dy - rP\{X \ge r\} \\ &= 0.4\sigma_X \int_{y_0}^\infty y e^{-0.5y^2} dy + \mu_X P\{X \ge r\} - rP\{X \ge r\} \end{aligned}$$

$$= 0.4\sigma_X \int_{y_0}^{\infty} y e^{-0.5y^2} dy + P\{X \ge r\}(\mu_X - r)$$

$$= 0.4\sigma_X e^{-0.5[(r-\mu_X)/\sigma_X]^2} + P\{X \ge r\}(\mu_X - r), \quad \text{(A3-7)}$$

which represents an exact value. By equality sense of formula (A3-5), then as an upper limit is

$$E\{S\} = \sigma_X(0.4e^{-0.5\lambda_\varepsilon^2} - \varepsilon\lambda_\varepsilon),\ -3 \le \lambda_\varepsilon \le 3. \quad \text{(A3-8)}$$

Thus, $E\{S\}$ becomes function of ε. The term $0.4e^{-0.5\lambda_\varepsilon^2} - \varepsilon\lambda_\varepsilon$ plays as a multiplier for standard deviation. Since $E\{S\}$ decays as λ_ε augments ($\lambda_\varepsilon \cong 3 \leftrightarrow \varepsilon \cong 0 \Rightarrow E\{S\} \cong 0$) and vice versa ($\lambda_\varepsilon \cong -3 \leftrightarrow \varepsilon \cong 1 \Rightarrow E\{S\} = max$). Setting $\lambda_\varepsilon > 3$ or $\lambda_\varepsilon < -3$ isn't acceptable. It depends on system conditions. For instance if the system is stable, i.e. $r \ge E\{X\}$, then $0 \le \varepsilon \le 0.5$.

A3.3. Asymmetric demand

Other continuous demand distributions such as lognormal can be fit relative to normal distribution. Hence, a lower limit on r can be estimated as

$$r \ge \mu_X + \delta\lambda_\varepsilon\sigma_X,\ -3 \le \lambda_\varepsilon \le 3. \quad \text{(A3-9)}$$

Where $\delta > 1$ for negative asymmetry and $\delta < 1$ for positive asymmetry. This value applies correction for standard deviation multiplier in both r and $E\{S\}$. The author proposes that $\delta \cong median_X / \mu_X$ suites nonlinear distributions that have one local maximum especially if they are moderately asymmetrical. Notice that the ratio $\mu_X / median_X$ is sometimes used as a measure of asymmetry (see Businger and Read 1999). Hence, formula (A3-7) modifies to

$$E\{S\} = \delta\sigma_X(0.4e^{-0.5\lambda_\varepsilon^2} - \varepsilon\lambda_\varepsilon),\ -3 \le \lambda_\varepsilon \le 3. \quad \text{(A3-10)}$$

This approximation isn't amenable to distributions such as exponential, power, linear, discrete, or compound distributions, and necessitates repeating the fitting procedure.

A4. Extreme TOR models

First derivative, respect to r, of formula (5) is

$$\frac{\partial TC(k,r)}{\partial r} = \frac{0.5D[-g(D/k + E\{X\} - r)\int_r^{\infty} f(x)dx + (O + gE\{S\})]}{(D/k + E\{X\} - r)^2}, \quad \text{(A4-1)}$$

which equates to zero at optimal conditions and reviewing the second derivative. Then

$$0 = -\int_r^{\infty} f(x)dx + \frac{(O + gE\{S\})}{g(D/k + E\{X\} - r)}, \quad \text{(A4-2)}$$

by applying optimality conditions of k, then

$$0 = -\int_r^{\infty} f(x)dx + \frac{2(w+\alpha)p(D/k + E\{X\} - r)}{gD}, \quad \text{(A4-3)}$$

which is rearranged to solve for optimal r yielding formula (21). First derivative, respect to r, of formula (10) leads to

$$0 = \int_r^{\infty} f(x)dx - \frac{(O + gE\{S\})}{g(D/k + E\{X\} - r)}, \quad \text{(A4-4)}$$

by applying the optimality conditions of k, (see Appendix A2), then

$$0 = \int_r^{\infty} f(x)dx - \frac{2(w+\beta k^2)p(D/k + E\{X\} - r)}{gD}, \quad \text{(A4-5)}$$

which is rearranged to obtain formula (22).

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Nomenclatures and Terminologies

a, b : lower and upper limits of the uniform distribution;
α : dimensionless, expected value-loss rate (frequency of reduction), $0 \le \alpha$;
β : dimensionless, dynamic value-return rate (per unit return), $0 \le \beta \le 1$;
C_α : loss component, added to the cost function, due to α;
χ : random variable representing unit value-loss rate, $0 \le \chi \le 1$;
ω, γ : regression constants of power trend between σ_X and D, $0 \le \omega$, $0 \le \gamma \le 1$;
D : average annual demand;
δ : dimensionless, correction factor for demand normality bias, $\delta \in \Re$;
$E\{S\}$: expected cycle shortage (in number of units), also μ_S;
$E\{X\}$: expected demand during lead time, also μ_X;
ε : accepted probability of cyclic shortage, a '*service level measure*' (SLM);
$f(x)$: probability density function of demand during lead time;
g : unit shortage cost per annum;
η : system profitability, $\eta \in \Re$;
η^* : optimum system profitability;
k : turnover rate (TOR)—positive value; here, it is dimensionless, $k > 0$;
k^* : optimum TOR;
λ_ε : dimensionless, standard normal multiplier corresponding to ε, $0 \le \lambda_\varepsilon \le 3$;
n : maximum number of times the system may be subjected to value-loss risk;
O : order cost;
p : unit value (purchase or production cost);
P_β : dynamic return component, added to the return function, due to β;
P_π : normal return component;
π : dimensionless, normal value-return rate (per unit return), $0 \le \pi$;
Q : order quantity;
r : constant review inventory level;
S : random variable representing cycle shortage;
σ_X : standard deviation of demand during lead time;
TC : average annual inventory cost;
u, v : regression constants of power trend between TOR and D, $0 \le u$, $0 \le v \le 1$;
w : holding cost rate; i.e., annual unit holding cost is wp;
X : continuous random variable representing demand during lead time;
ξ : random variable representing value-loss state $\xi \in \{0, 1\}$ with $\{f_\xi(0), f_\xi(1)\}$;
Z : average annual profit (net return).