Mansoura Engineering Journal, (MEJ), Vol. 22, No. 3, September 1997 C. 45

APPLICATION OF FINITE ELEMENT RESIDUAL SCHEME TO STEADY UNCONFINED FLOW SUBJECT TO RAINFALL

by

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دراسة السريان المستقر الغير محصور والمعرض للمطر باستخدام نظرية العناصر المحدودة

خلاصة :

في هذا البحث تم إستخدام نظرية العناصر المحدوده كنموذج عددى (نموذج التصرف الباقى) فى إتجاهين لحل مشكلة السريان الغير محصور خلال التربة المتجانسة والغير متجانسة وذلك فى وجود تساقط منتظم ومستقر .

تمت الدراسة فى وجود وسط محدد بمجرين مائيين متماثلى المستوى المائى أو غير متماثلين . ثم عمل مقارنة بين نتائج الدراسة بالنموذج العددى ونظائرها بإستخدام الحل الرياضى لمعادلة " بوسينسك " .

يتضح من النتائج أن الحل الرياضى لمعادلة بوسينسك يمكن تطبيقه فى الحالات التى يكون فيها معدل المطر أو الرى منسوبا إلى معامل النفاذية أصغر من ٠,٢ (R/k < 2) ، اما فى حالة (R/k > 2) فالنتائج تعطى انحرافا كبيرا إذا ما قورنت بمثيلتها فى الحل العددى . فى حالة التربة الغير متجانسه فالنتائج عامه تتأثر بكل من خصائص السريان ومناسيب المياه الباطنيه

ABSTRACT

A two- dimensional finite element method is used to study the problem of steady-state unconfined seepage due to rainfall through porous media. The method is based on the finite element residnal scheme. Homogeneous and nonhomogeneous soil domains are considered and solved numerically. Numerical results are compared with that obtained from the analytical approximations to be in the same order of magnitude. To facilitate its practical use, the scheme has been programmed for computer solution using FORTRAN-IV language.

INTRODUCTION

Subsurface drainage system aims at maintaining the water table at optimum depth of root zone. The water table fluctuations in response to replenishment and deep percolation is common in many groundwater basins and affects either the entire basin or parts of it. Groundwater may be replenished by

Accepted, July, 19, 1997.

natural precipitation, irrigation, or artificial recharge. Most of the existing theories attempt to describe the water table fluctuations in response to constant replenishment between equally-spaced drains on flat lands.

Boussinesq [1] derived a nonlinear partial differential equation for groundwater flow in a gently sloping aquifer above an impermeable layer. The Boussinesq equation is based on the Dupuit-Forchheimer assumptions and is an approximate model used to describe the phreatic surface. Warner [9] solved the Boussinesq equation, by taking replenishment into account, for groundwater flow in a phreatic aquifer resting on a sloping bed between two reservoirs and between a water divide and a reservoir.

Kraijenhoff Van de Lear [5,6] analyzed the effects of constant recharge and intermittent recharge on groundwater flow toward drains in a horizontal aquifer using the Boussinesq equation and applying the instantaneous unit hydrograph concept. Massland [7] also solved the Boussinesq equation for a horizontal aquifer receiving both constant recharge and intermittent recharge. Schmid and Luthin [8] analyzed the problem of a steady-state drainage of sloping land with constant recharge rate by parallel ditches resting on an impermeable layer by solving the Boussinesq equation.

The purpose of this paper is: 1) to have a numerical practical tool to handle steady unconfined seepage flow which is affected by rainfall, or artificial recharge; 2) to solve more complex problems that are not easy to be solved analytically; and 3) to compare the obtained results with the closed form solution for Boussinesq's equation considering steady unconfined seepage between symmetrical and unsymmetrical water ways. In addition, the case of the multiaquifer (double layers) is demonstrated.

MATHEMATICAL FORMULATION

Consider the problem of two-dimensional unconfined aquifer shown in Fig. (1). Assuming that the soil matrix and the fluid in the pores are incompressible. The continuity equation for steady flow through a saturated, homogeneous and isotropic porous medium describing this problem, in a cartesian reference frame is the Boussinesq equation, Harr[3]:

$$\frac{d}{dx}\left[kh \frac{dh}{dx} \right] + R = 0 \qquad (1)$$

in which k is the hydraulic conductivity, LT^{-1}; R is a constant replenishment rate, LT^{-1}; and h is the hydraulic head, L.

Eq.(1) is to be solved subject to the following boundary conditions:

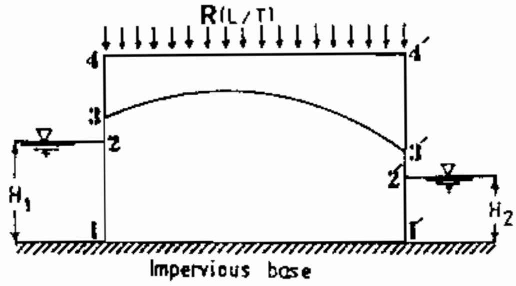

Fig.(1) Phreatic aquifer subject to rainfall, R.

$\partial h/\partial y = 0$ on the impervious boundary 1-1';

$h = H_1$ on the upstream face 1-2;

$h = H_2$ on the downstream face 1'-2';

$h = y$ on the upstream and downstream seepage faces 2-3 and 2'-3', respectively;

$\partial h/\partial y = R/k$ and $h = y$ on the free surface 3-3'; and

$k\ \partial h/\partial y = R$ on the ground boundary 4-4'.

RESIDUAL FLOW TECHNIQUE

In this technique, the geometry of the finite element mesh is kept constant during the iterative solution process. The approach proposed by Desai [2], is the so-called Residual Flow procedure. In the case of steady-state problems, the iterative solution process is initiated by performing a confined analysis in which boundary conditions of "previous" and "impervious" types only are applied to the various portions of the mesh contour. The nodal hydraulic heads obtained by this calculation are used to find the elements crossed by line h=y(x) (denoted by F-S in Fig.(2)), that represents a first approximation of the free surface. The position of this line (on which only the conditiun h=y is fulfilled) within the elements is determined by means of:

$$h = N^T \{ H \}^{e} \qquad (2)$$

where N is the vector of interpolation functions.

In order to enforce also the second boundary condition $\partial h/\partial y = R/k$, it is necessary to evaluate the flow velocity V that crosses the free surface, FS, taking into account Darcy's law.

$$V_x = -k_x\ \partial h/\partial x \quad \text{and} \qquad (3)$$
$$V_y = -k_y\ \partial h/\partial y$$

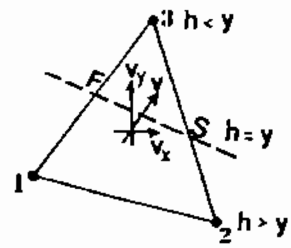

Fig.(2) Approximation of free surface, FS, through an element.

to satisfy the second boundary condition, the vertical velocity component must have the same value of R.

The flux (associated with V) crossing the free surface can be obtained, using a terminology suited for stress analysis problems, as a "distributed load" on segment, FS. This load has to be reduced to a certain value in order to fulfill the second boundary condition. This can be done by applying to the nodes of the corresponding element a set of "residual" nodal flux q that produce through the segment, FS, a flux equal to the one associated with V, but with opposite sign if $V_y>R$.

$$q = + \int_{FS} N\ V\ T\ ds \qquad (4)$$

In Eq. (4), T represents the "thickness" of the finite element discretization. By applying the residual flux q , evaluated for all the elements crossed by the free surface, to the relevant nodes of the mesh and by solving Eq. (5), a new vector of nodal hydraulic heads is determined.

$$[\ K\ ]\ \{\ H\ \} = \{\ q\ \} \qquad (5)$$

where [K] is the total flow matrix. This will lead to a different, more refined, approximation of the free surface geometry and to a new residual flow vector. The iterative process is continued until the changes in geometry of the free surface become negligible.

ANALYTICAL APPROXIMATION

In order to provide a simple check of the numerical model, and to compare its results on an order of magnitude scale, analytical approximation is presented. Kamensky obtained his approximate solution to describe the free surface by integrating Eq. (1) which takes the form:

$$h = \sqrt{h_1^2 - (h_1^2 - h_2^2)/L + (R/k) (L - x) x} \qquad (6)$$

Describing the distance to the maximum elevation of the free surface in Fig. (3) as a called the water divide, it is found from Kamensky [4] that:

$$a = L/2 - k/R - (h_1^2 - h_2^2)/2L \qquad (7)$$

where a is the distance measured from point o as shown in Fig.(3).

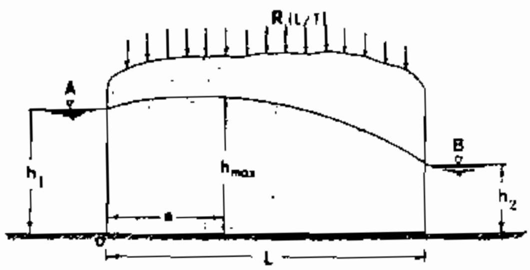

Fig.(3) Flow domain under consideration.

NUMERICAL RESULTS

It is necessary to mention here that, the residual flow procedure is one of the constant mesh methods which require some specific provisions when the intersection between the free surface and the seepage face (i.e., the "exit" point of the free surface) has to be determined. To do this, exit points are initially located in correspondence with the last nodal points of the adopted schematization and then they were moved step by step upward until the pore water pressure in each node above them resulted negative. In fact, if we assume at the beginning of an iteration the position of the exit points those are above the real ones, the results of that iteration will not allow us to determine their new location for the next iteration.

The problems under consideration have been solved numerically under the following assumptions:

i) The replenishment is received at a constant rate which is less than the hydraulic conductivity of the soil;
ii) The deep percolation moves vertically downwards until it joins the main groundwater body, and
iii) The soil is drained by parallel drains which may have either the same water levels or different water levels.

To solve these problems numerically, the assumed domain has been discretized as shown in Fig.(4) using 286, three nodded triangular elements with a total of 174 nodal points. Referring to the first type of the problems, assuming R/k=0.2, $H_1=H_2=2.0$ m, the obtained water table location after three iterations has been illustrated in Fig.(5) compared with the analytical one. It can be seen that the present solution for the free surface agrees with the approximate one everywhere except in the vicinity of the seepage faces.

As a result of the present solution, flow net, outflow velocities (V_e), and the floor creep velocity are illustrated as shown in Fig.(6). From this figure it is observed that the maximum exit velocities occur at the drains water levels and their minimum values occur at the impermeable layer face. Also, one can see that, at the midway of the domain, a stagnation point, (V=0) has been developed which agrees with the fact that, at the line of symmetry the flow has been divided and no flow takes place perpendicular to this line. Several problems have been solved considering different values of R/k (0.05-0.5), and the obtained results of the height of exit points are illustrated in Fig.(7).

Referring to the solution of the second type of the problems, the same value of R/k=0.2 has been used again with $H_1=2H_2=4$ m. The free surface location, that was obtained after three iterations too, has been illustrated in Fig.(8). compared with the approximate one. The corresponding flow net, exit velocities , and the floor creep velocity have been illustrated in Fig.(9).

To define the range of the approximate solution validity, some additional problems were solved considering R/k= 0.4-0.8. Obtained phreatic surfaces are illustrated in Fig.(10). From this figure, it is clear that the difference between the present solutions and the approximate ones is increased with as the ratio of rainfall to permeability, R/k. is increased.

To illustrate the flow behaviour in the case of heavy rainfall rate within upstream water head that is enough to overcome a part of the upstream outflow, a problem with retaining eight meters of water in the upstream side with the downstream side dry, has been solved with R/k=0.6. Obtained free surface location with the corresponding flow net, inflow-outflow velocities, and the floor creep velocity are illustrated in Fig.(11). From this figure it is noted that, the point of stagnant velocity has been developed at the upstream inlet face and the flow has been divided to inflow

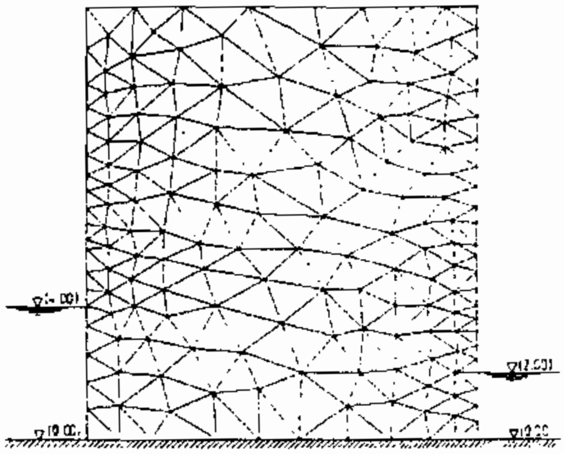

Fig.(4) Finite elements grid system of the domain.

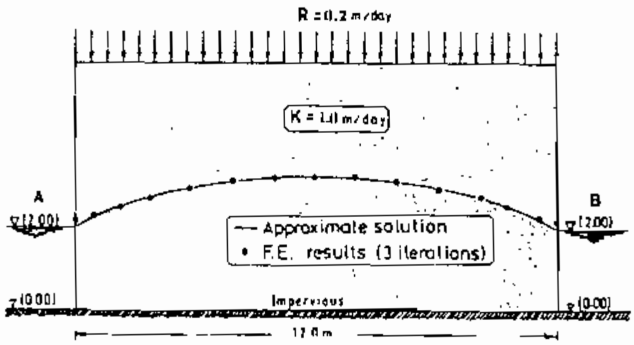

Fig.(5) Comparison between analytical and numerical solutions.

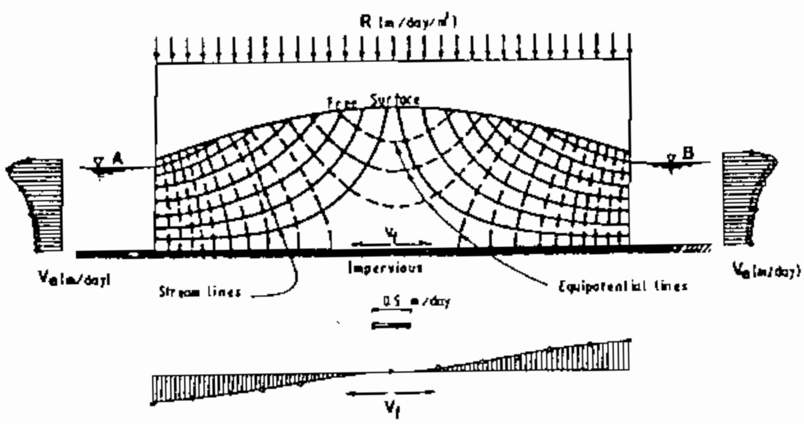

Fig.(6) Flow net, outlet and creep velocities with R/k=0.2.

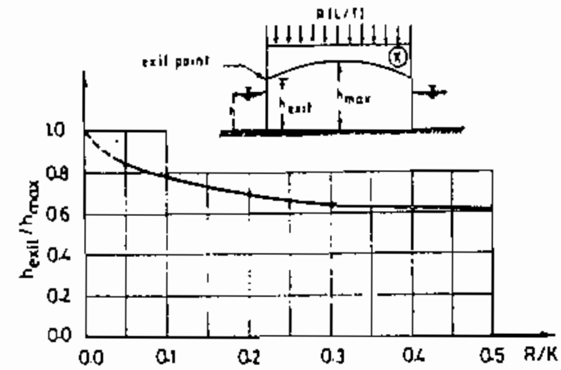

Fig.(7) Relative height of exit points with the variation of R/k.

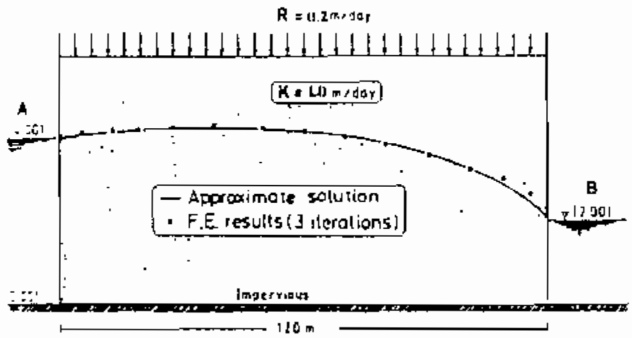

Fig.(8) Comparison between analytical and numerical solutions for different water levels.

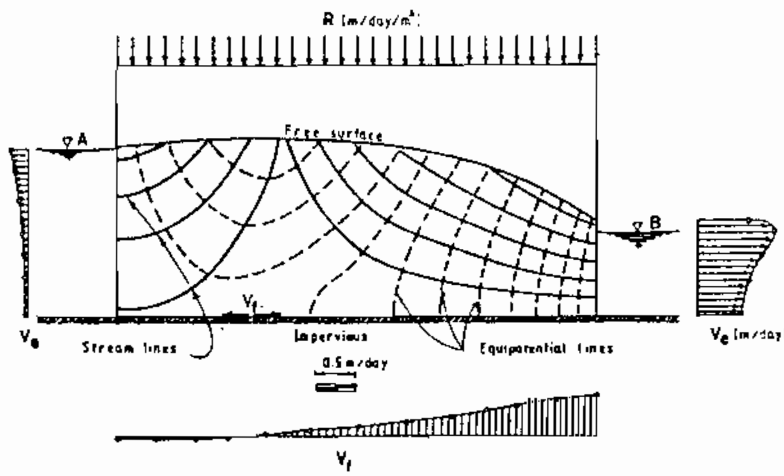

Fig.(9) Flow net, outlet and creep velocities for different water levels with R/k=0.2.

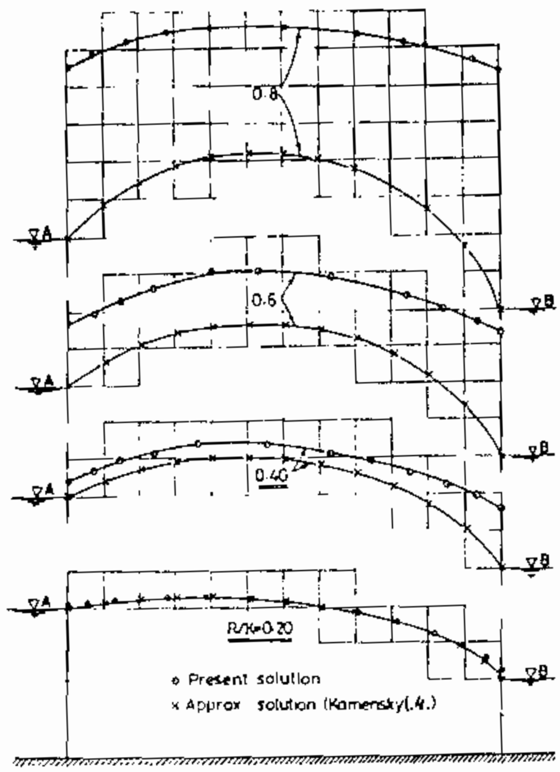

Fig.(10) Obtained water tables for different values of R/k comparing with the analyticalsolution.

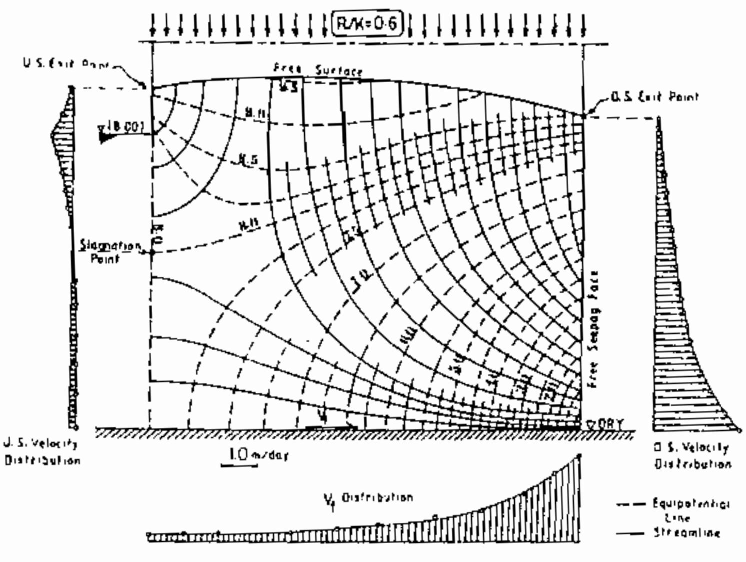

Fig.(11) Flow net, outlet and creep velocities for R/k=0.6.

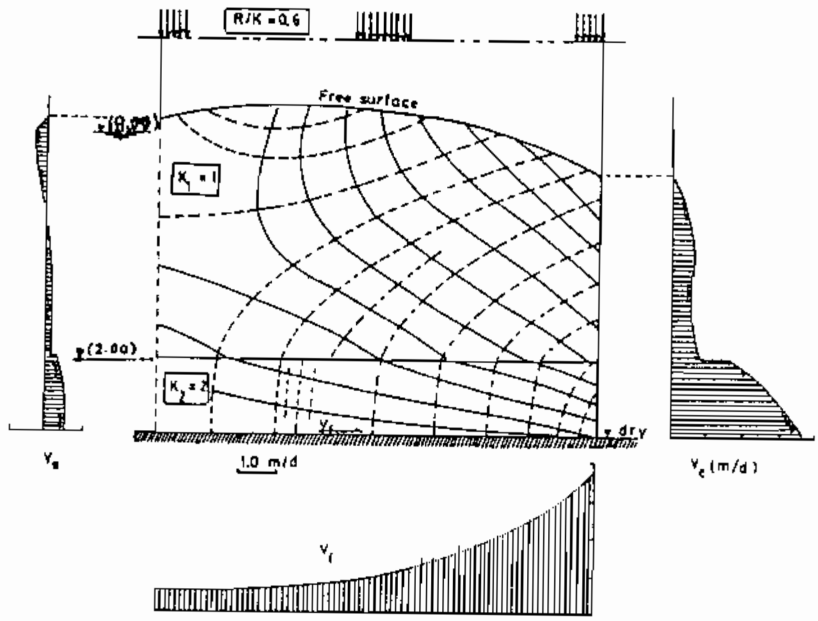

Fig.(12) Solution of the problem for nonhomogenous medium $k_1/k_2=1/2$ with R/k=0.6.

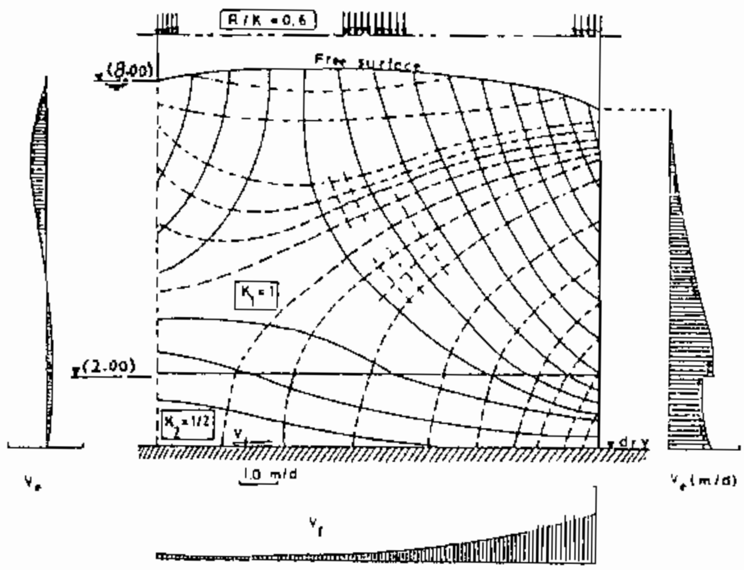

Fig.(13) Solution of the problem for nonhomogenous medium $k_1/k_2=2.0$ with R/k=0.6.

and outflow under and upper this point, respectively.

The problem of steady unconfined seepage subject to rainfall through nonhomogeneous medium has been solved with R/k=0.6. To handle this type of problems, two cases are considered. One of them has a lower layer with a higher permeability than the upper one, $k_1/k_2=1/2$. The other case has a higher permeability in the upper layer than the lower one, $k_1/k_2=2.0$. The obtained results are illustrated in Figures (12) and (13), respectively. Inlet, outlet and creep velocities are plotted with the corresponding flow net. Frum these figures it is observed that, decreasing the lower layer permeability rises the level of water table and vice-versa.

CONCLUSIONS

A numerical mudel for the prediction of the free surface position, taking into account the influence of steady rainfall, is developed. The model is based on the finite element residual scheme. Ohtained results are compared with the analytical solution developed by Kamensky based on Boussinesq's equation. From these analyses, it may be concluded that, Boussinesq's equation is based on the Dupuit-Forchheimer assumptions, which are not well matching the actual flow system. This limitation, therefore, introduces some error in the analytical solutions. The error would be greater near the boundary faces than at the midpoint for light rates uf rainfall, $(R/k<0.2)$. In the cases of heavy rates $(R/k>0.2)$, the analytical results largely deviates from the numerical results.

Generally, for the case of light rates of rainfall, the analytical solution may be used with a reasonable accuracy to predict the phreatic surface between drains but not to predict the level at which water seeps out from the region. Steady unconfined flow through a two-layered aquifer system with the presence of steady rainfall has been solved numerically using the residual flow finite element technique. The non-homogeniety of the system affects both the flow patern and water table levels.

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