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ON THE THEORY OF FIELD PATTERN SYNTHESIS

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ABSTRACT

Uniform array synthesis is treated as a problem of approximation in a Hilbert space. Approximations in the mean square sense and in the Sobolev sense are considered. Following Tichonov's regularization technique for solving inverse problems, the array excitation current is subjected to a certain constraint in order to make the design less-sensitive to construction errors. A variational formulation of the problem using Lagrange multiplier method is presented. The equation determining the excitation current is obtained in a general operator form and is then put in a form amenable to numerical computation. Existence and uniqueness of solution are proved. Some example problems are presented.

I- INTRODUCTION

Several techniques have been developed for the synthesis of uniform linear arrays that will produce a prescribed field pattern within specified error limits. These involve: Fourier series methods, Dolf - Chebyshev arrays and related minimax techniques, methods based on interpolation theory or eigenfunction expansions, and iterative sampling techniques [1] - [11] . A review of the development of synthesis theory has been given by Feld and Bakhrakh [12] and is also presented in many books of which [13] - [15] are typical.

The usefulness of any of the above methods ultimately relies on the question of realization. Realization will be possible if the required complexity and accuracy in the excitation current is kept low. In practice, there is always unavodiable inaccuracy and errors in the antenna current and the actual pattern will differ from the theoretical. These errors may be either predictable (deterministic) or random. The effect of the first type on the radiation pattern can be calculated by classical methods and may be taken into account in the design procedure. Some sources of random errors are: accidental deviation of the amplitudes or phases of the element currents from their design values, translational errors in element locations and missing elements.

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Although these errors may be small, they are ever present and may cause pronounced deviation of the synthesized pattern from the desired one. This deviation depends on, how much the radiation pattern is "Sensitive" to smallerrors in the excitation currents or element locations. Some theoretical solutions to the synthesis problem may be completely worthless because of excessive sensitivity: they are unrealizable. in a practical sense.

The sensitivity and stability of solutions to the pattern synthesis problem have been discussed by Deschamps and Cabayen [16] . They characterized the synthesis problem as an example of "Improperly posed" problems. This is a class of problems of mathematical physics that is encountered in many fields, e.g., remote sensing, target identification, pattern recognition,... etc. Earlier contributions to the study of these problems are due to Tikhonov and Lavrantev [17] - [18] . The synthesis technique presented in this paper makes use of the results of these authors. The optimum array excitation is obtained through constrained minimization of certain objective function (error criterion). The later comprises weighted difference between the desired pattern and the synthesized one. The choice of the error criterion is discussed in the next section.

II- ERROR CRITERION

The synthesis problem considered in this paper may be stated as follows. Given a uniform linear array of n-elements. It is required to find a set of excitation currents I so that the resulting array factor P approximates a desired field pattern P_d. It will be assumed that P_d is a piecewise continuous function defined over the interval $-1 \leqslant u \leqslant 1$, where $u = \cos\theta$ and θ is the angle from broadside. Excitation currents $I \equiv (I_1, I_2, \ldots, I_n)$ will be treated as points in an n-dimensional space $\mathcal{J}$ and array factors P as elements of the space $\mathcal{P}$ of square integrable functions defined over the interval $[-1, 1]$. Excitation currents and array factors are related by the transformation

$$P = \hat{F} I \qquad \ldots\ldots\ldots(1)$$

The operator $\hat{F}$, as known from array theory, is linear and bounded and, therefore, continuous [19] . It is defined by a scalar product of the form [13] .

$$\hat{F} I = (I, \Psi) = \sum I_n \Psi_n \qquad \ldots\ldots\ldots(2)$$

where Ψ_n are in general exponential functions of the form exp (jnkdu), where "d" is the interelement spacing.

To solve the synthesis problem means to find an excitation I such that $\hat{F}$ I is close to P_d. In order to give a precise definition of "Closeness" of two elements in the space $\mathcal{I}$ or $\mathcal{P}$ we begin with defining norms $\|I\|$ and $\|P\|$ and letting the distance between two elements be the norm of their difference. For example, the distance between P and P_d in the space $\mathcal{P}$ is given by

$$D = \|P - P_d\| \qquad \text{........(3)}$$

D measures the deviation of the pattern P from the desired one and will be called "error criterion".

The spaces $\mathcal{I}$ and $\mathcal{P}$ so organized become metric spaces. Norms and distances satisfy a number of properties that can be found in [19] .

There are several ways for defining norms, but only those norms which make $\mathcal{I}$ or $\mathcal{P}$ into a Hilbert space are particulary important. This is because in Hilbert spaces one can define angles, perpendiculars and projections and perform most geometric constructions familiar in Euclidean spaces. In fact, Hilbert spaces are in essence natural reilization to the latter in the realm of functional spaces.

The norms $\|P\|_2$ and $\|I\|_2$ defined, respectively, by the scalar products

$$\|P\|_2 = (P, P)^{0.5} = \left(\int_{-1}^{1} \|P\|^2 \; w(u) \; du\right)^{0.5} \qquad \text{.......(4)}$$

and

$$\|I\|_2 = (I, I)^{0.5} = \left(\sum_{m=1}^{n} |I_m|^2\right)^{0.5} \qquad \text{........(5)}$$

are ones for which $\mathcal{P}$ and $\mathcal{I}$ are Hilbert spaces. Distance D in this case is defined by

$$D = \|P - P_d\|_2 = \left(\int_{-1}^{1} (P - P_d)^2 \; w(u) \; du\right)^{0.5} \qquad \text{........(6)}$$

It represent the integral of the squares of deviations of P from P_d weighted by the function w(u) which reflects the relative accuracy with which P approximates P_d on the different parts of the interval $-1 \leqslant U \leqslant 1$.

There are other norms which makes a space $\mathcal{P}$ into Hilbert space. An example is the norm $\|P\|_s$ defined by

$$\|P\|_s^2 = (P, P) + (P', P') = \int_{-1}^{1} |P|^2 \; w_1(u) \; du + \int_{-1}^{1} |P'|^2 \; w_2(u) \; du \qquad \text{..........(7)}$$

where the prime denotes differentiation with respect to u, and w_1 and w_2 are weighting functions.

Hilbert spaces having norms of the form (7) or its generalizations are now commonly called Sobolev spaces, after L. Sobolev who developed their theory [20]. Closeness of two elements P and P_d according to norm (7) ensures that, not only values of P and P_d are close, but also the values of their derivatives P and P_d.

Having chosen a specific norm, the problem now is to find the current $I \equiv (I_1, I_2, \ldots, I_n)$ which minimizes the error $E = D^2$. We show that this amounts to solving a set of linear equations. Consider E as function of I and let I be increased by a small increment I. Then from (1), (3), (4) we get

$$\begin{aligned} E(I + \Delta I) &= \|P - P_d\|^2 \\ &= \|\hat{F}(I + \Delta I) - P_d\|^2 = \\ &= (\hat{F}(I + \Delta I) - P_d, \hat{F}(I + \Delta I) - P_d) \quad \ldots\ldots(8) \end{aligned}$$

Using known properties of linear operators and scalar products, we find that

$$\begin{aligned} E(I + \Delta I) &= (\hat{F}I - P_d, \hat{F}I - P_d) + (\hat{F}\Delta I, \hat{F}I - P_d) \\ &\quad + (\hat{F}I - P_d, \hat{F}\Delta I) + (\hat{F}\Delta I, \hat{F}\Delta I) \quad \ldots\ldots(9) \end{aligned}$$

Hence upon neglecting second order terms we get

$$\begin{aligned} \Delta E(I) = E(I + \Delta I) - E(I) &= (\hat{F}\Delta I, \hat{F}I - P_d) + \\ &+ (\hat{F}I - P_d, \hat{F}\Delta I) \quad \ldots\ldots(10) \end{aligned}$$

The adjoint operator $\hat{F}^+$ is defined by

$$(\hat{F}I, P) = (I, \hat{F}^+ P) \quad \ldots\ldots\ldots(11)$$

It follows that

$$\Delta E(I) = (\Delta I, \hat{F}^+ \hat{F}I - \hat{F}^+ P_d) + (\hat{F}^+ \hat{F}I - \hat{F}^+ P_d, \Delta I) \quad \ldots\ldots\ldots(12)$$

If E(I) is a minimum, then $\Delta E(I)$ should be zero for any small increment ΔI. From (12), this requirement implies

$$\hat{F}^+ \hat{F}I - \hat{F}^+ P_d = 0$$

or

$$\hat{F}^+ \hat{F}I = \hat{F}^+ P_d \quad \ldots\ldots\ldots(13)$$

Let us determine the explicit form of $\hat{F}^+$ and $\hat{F}^+ \hat{F}$. We have from the definitions of the scalar product and the operator F (equation (2)),

$$(\hat{F}I, P_d) = \int_{-1}^{1} (I, \psi) P_d \; w(u) \, du$$

$$= (I, \int_{-1}^{1} \psi P_d \; w(u) \; du)$$

$$= \sum_{m=1}^{n} I_m \int_{-1}^{1} \psi_m P_d \quad w(u) \, du = (I, \hat{F}^+ P_d) \quad \ldots\ldots\ldots(14)$$

Hence, we see that $\hat{F}^+ P_d$ is a column vector B having n-components given by

$$B_i = (\hat{F}^+ P_d)_i = (\psi_i, P_d) \quad i = 1, 2, \ldots\ldots, n \quad \ldots\ldots\ldots(15)$$

i.e. $\hat{F}^+ P_d$ is the projection of P_d on the set ψ. Also, $\hat{F}^+ \hat{F}I$ is a vector whose i <u>th</u> component is

$$(\hat{F}^+ \hat{F}I)_i = (\psi_i, \hat{F}I) = \sum_j I_j (\psi_i, \psi_j) \quad \ldots\ldots\ldots(16)$$

It is then easily seen that $\hat{F}^+ \hat{F}$ is a matrix operator given by

$$\hat{F}^+ \hat{F} = \hat{A} = a_{ij} = \left[(\psi_i, \psi_j)\right] \quad \ldots\ldots\ldots(17)$$

$$i = 1, 2, \ldots, n \qquad j = 1, 2, \ldots\ldots, n$$

Equation (13) is therefore equivalent to the system

$$\sum_{j=1}^{n} I_j (\psi_i, \psi_j) = (\psi_i, P_d), i = 1, 2, \ldots\ldots, n \quad \ldots\ldots\ldots(18)$$

or, in matrix form

$$\hat{A} I = \hat{B} \quad \ldots\ldots\ldots(19)$$

Matrix $\hat{A}$ is positive definite. It is the matrix of the quadratic form:

$$\int_{-1}^{1} (\psi_1 e_1 + \psi_2 e_2 + \ldots\ldots + \psi_n e_n)^2 \times w(u) \, du$$

Where $e_1, e_2, \ldots, e_n$ are unit vectors, Moreorer, if the functions are linearly independent, then $|A| \neq 0$ and A is non-singular. This is a necessary and sufficient condition for the existence of a unique solution to the system (19) and, hence, to the synthesis problem.

The above results have been obtained for approximation according to $\|P\|_2$ norm. In case of Sobolev norm $\|P\|_s$, similar results can be obtained. The required current is the solution to the system.

$$\hat{A}_s I = \hat{B}_s \qquad \ldots\ldots\ldots(20)$$

Where $\hat{A}_s$ is a square n xn matrix given by

$$\hat{A}_s = \left[a^s_{ij}\right] = \left[(\psi_i, \psi_j) + (\psi'_i, \psi'_j)\right] \qquad \ldots\ldots\ldots(21)$$

$$i, j = 1, 2, \ldots, n$$

and $\hat{B}_s$ is the column. vector:

$$\hat{B}_s = \left[(\psi_i, P) + (\psi'_i, P'_d)\right] \qquad \ldots\ldots\ldots(22)$$

$$i = 1, 2, \ldots, n$$

As an illustrative example, consider the design of an 11-element center symmetric linear array with half-wave interelement spacing, (Fig. (1)). The desired pattern is a sector beam defined by:

$$P_d = 1 \quad -0.5 \leqslant u \leqslant 0.5$$
$$= 0 \quad |u| > 0.5 \qquad \ldots\ldots\ldots(23)$$

such a pattern is of special importance in some applications such as navigational radars and warning approach systems.

The operator equation (1) relating the field pattern and the current takes the form:

$$P = \hat{F} I = \sum_n I_n \psi_n = I_o + 2\sum_{m=1}^{4} I_m \cos m\pi u \qquad \ldots.(24)$$

$$\text{i.e.,}\ \psi_m = \epsilon_m \cos m\pi u, \quad \epsilon_m = 1 \quad m = 0$$

$$= 2 \quad m > 0$$

Two error criteria have been considered: the mean square or Gaussian error, where approximation is carried out according to the $\|P\|_2$ norm, and the error criterion corresponding to the sobolev norm $\|P\|_s$. In each case, calculations have been made for the following weighting functions:

(i) a constant $w(u) = 1 \quad -1 \leqslant u \leqslant 1$

(ii) a staircase function $w(u) = 1 \quad |u| < 0.5$

$= 2 \quad 0.5 < |u| < 1$

The latter is used to emphasize errors in the sidelobe region.

The excitation currents in each case are obtained by solving the pertinent system of linear equations:

$\hat{A}I = \hat{B}$ or $\hat{A}_s I = \hat{B}_s$. The elements of the corresponding matrices are scalar products comprising integrals of products of trigonometric functions and are easily computed. With a weighting function equal to a constant, the computation is further simplified, since the functions $\{\psi_n\}$ and $\{\psi'_n\}$ form orthogonal sets with respect to the norms $\|P\|_2$ and $\|P\|_s$. The matrices $\hat{A}$, $\hat{B}$, $\hat{A}_s$, $\hat{B}_s$ are diagonal. In case of mean square error criterion, the excitation currents have a simple interpretation, they are the coefficients of the Fourier - Cosine expansion of the desired pattern. This interpretation is true only for the special case of half-wave interelement spacing and constant weighting function. Also, it is worthnoting that for this particular case the approximations according to $\|P\|_2$ and $\|P\|_s$ norms coincide. Thus, the following relations can be easily proved:

$$\hat{A}_s = \hat{C}\hat{A}, \quad \hat{B}_s = \hat{C}\hat{B} \qquad \text{.........(25)}$$

where $\hat{C}$ is a diagonal matrix defined by;

$$\hat{C} = [C_{ij}], \qquad \text{.........(26)}$$

$$C_{ij} = 0 \qquad i \neq j$$

$$= I + \epsilon_i^2 \; i^2 \pi^2 \qquad i = j = 0, 1, \ldots\ldots, n$$

It then follows that the systems $\hat{A}I = \hat{B}$ and $\hat{A}_s I = \hat{B}_s$ are equivalent. Hence the two methods of approximation give identical results. This is an important property of the Fourier expansion: the series expansion not only approximates the given function in the mean square sense, but also the derivative of the expansion is close to the derivative of this function in the same sense [20]. This property gives the Fourier expansion method some advantage over other synthesis techniques.

For the other choice of the weighting functions, the two norms $\|P\|_2$ and $\|P\|_s$ are not topologically equivalent and the two methods of approximation yield different results as shown from curves c,d on Fig.(2). The gaussian error criterion provides a better match between the amplitudes of the desired and synthesised patterns, while the sobolev norm results in a better match between their derivatives. In both cases, there is a general agreement between the desired pattern and the approximate one, but the quality of approximation is not the same over the whole interval. This is because the error criteria used are average error measures: they do not specify the excursion of the approximation function at any arbitrary point. The quality of approximation is degraded near the discontinuity of the desired pattern, there relatively large oscillations are observed (Gibbs - phenomenon). The use of a weighting function results in some improvement, near the discontinuity, but at the expense of larger deviation in other parts of the interval as seen by comparing curves b, c of Fig. (2).

In some applications, it is necessary to limit the maximum excursion of the approximating function from the desired pattern. This can be done by using a Chebyshev-type error criterion. In a recent publication, however, it has been shown that such an approximation can be achieved through a series of weighted least square approximations [21]. The methods of the present work would be useful in this context.

III- VARIATIONAL FORMULATION

In the previous section it has been shown that the functional $\|\hat{F}I - P_d\|$ has a unique absolute minimum. From the point of view of realization, however, we are not as much interested in minimizing the absolute error $\|\hat{F}I - P_d\|$ as in determining that excitation I such that the performance $\hat{F}I$ will not deteriorate if $\|I\|$ is subjected to small random errors. In other words, we may accept more error, if necessary, in order to reduce the sensitivity of the design to an acceptable level.

A measure of the array sensitivity to random errors is the norm I of the excitation current [13, 23]. The synthesis problem may then be restated as follows.

Given the geometry of the array, the desired field pattern P_d, and the allowable deviation E of the synthesized pattern P from P_d,

$$\|P - P_d\| = \|FI - P_d\| \leqslant E \qquad \text{.........(27)}$$

It is required to determine the excitation current I having minimum norm which satisfy requirement (27).
To solve this variational problem, we have to minimize the functional

$$M(I) = \|I\|^2 + \alpha \|\hat{F}I - P_d\|^2 \qquad \text{.........(28)}$$

where α is a positive Lagrangian multiplier. By the reciprocity theorem of the calculus of variation, minimizing the functional M is equivalent to minimizing the functional:

$$M(I) = \alpha \| I \|^2 + \| \hat{F}I - P_d \|^2 \qquad \text{.........(29)}$$

Minimization of such a functional is the basis of the regularization method suggested by Tikhonov [18] for solving improperly posed problems of mathematical physics, in which the question of stability and sensitivity are an issue.

In addition to keeping I a minimum, we may have some further requirements. Thus, we may specify the field in certain directions by equalities of the form:

$$\hat{F}_m I = P(u_m) = 1 \qquad \text{.........(30)}$$

$$\hat{F}_o I = P(u_o) = 0 \qquad \text{.........(31)}$$

where u_m defines the main beam direction and u_o the null of the main beam. The first equality is a normalization constraint while the latter specifies the main beam width.

The problem then is to minimize $\|I\|^2$ over the set:

$$S \equiv \left\{ I : \| \hat{F}I - P_d \|^2 \leqslant E, \ \hat{F}_m I = 1, \ \hat{F}_o I = 0 \right\} \quad \text{...(32)}$$

The functional M(I) now takes the form:

$$M(I) = \alpha \| I \|^2 + \| FI - P_d \|^2 + \beta_1 \hat{F}_m I + \beta_2 \hat{F}_o I \quad \text{...(33)}$$

where β_1 and β_2 are additional Lagrangian multipliers.

Note that minimization of the norm $\|I\|^2$ not only improves the sensitivity of the array but also minimizes the ohmic losses in the radiators and, therefore, improves the gain of the antenna.

In appendix, it is proved that the set S is closed and convex. Also, it can be easily shown that the function $f(I) = \|I\|^2$ is strongly convex and twice continuously differentiable. It follows that this function has a unique minimum in S [24]. This proves the existence and uniqueness of solution of the synthesis problem in the present formulation. According to Lagrange multiplier rule (Kuhn-Tucker theorem), this minimum is obtained by writing that the differential of the functional M given by equation (33) with respect to I is zero.

From equation (12) of the previous section, we have:

$$\Delta \|FI - P_d\|^2 = (\hat{I}, \hat{F}^+ \hat{F}I - \hat{F}^+ P_d) + (\hat{F}^+ \hat{F}I - P_d, \Delta I) \quad \ldots.(34)$$

Also, from equations (30), (31) and the definition of the operator F in equation (2), we get

$$\Delta \hat{F}_m I = \Delta(I, \psi^m) = (I + \Delta I, \psi^m) - (I, \psi^m)$$
$$= (\Delta I, \psi^m) \quad \ldots\ldots\ldots(35)$$

and, similarly

$$\Delta \hat{F}_o I = \Delta(I, \psi^o) = (\Delta I, \psi^o) \quad \ldots\ldots\ldots(36)$$

where ψ^m denotes the value of ψ at $u = u^m$ the direction of the main beam, and ψ^o the value of ψ at $u = u_o$ - the first null of the desired pattern.
Finally, the differential of I^2 is given by

$$\Delta \|I\|^2 = \Delta(I, \Delta I) = (\Delta I, I) + (I, \Delta I) \quad \ldots\ldots\ldots(37)$$

From equations (33) - (37), we get:

$$\Delta M = (\Delta I, \alpha I + \hat{F}^+ \hat{F}I - \hat{F}^+ P_d + \beta_1 \psi^m + \beta_2 \psi^o)$$
$$+ (\alpha I, + \hat{F}^+ \hat{F}I - \hat{F}^+ P_d + \beta_1 \psi^m + \beta_2 \psi^o, \Delta I) \quad \ldots.(38)$$

The unknown multipliers β_1, β_2 in equation (38) differ from the corresponding ones in equation (33) by a factor of 0.5, which is not important at present. For M to be extremum, ΔM should be zero for any small ΔI, and from equation (38) this implies,

$$\alpha I + \hat{F}^+ \hat{F}I - \hat{F}^+ P_d + \beta_1 \psi^m + \beta_2 \psi^o = 0 \quad \ldots..(39)$$

This is the general operator equation determining the optimum excitation current. From the discussions of the previous section, the explicit forms of the operators $\hat{F}^+ \hat{F}$ and $\hat{F}^+$ depend on the particular norm according to which the synthesized pattern should approximate the desired one. For approximation in the mean square (guassion) error sense, we see from equation (17) that $\hat{F}^+ \hat{F}$ is the square matrix A given by:

$$\hat{A} = \left[(\psi_i, \psi_j)\right] \quad i,j = 1,\ldots., n \text{ (equation 17)}$$

and from equation (15), $\hat{F}^+ P_d$ is the column matrix:

$$\hat{B}_i = \left[(P_d, \psi_i)\right]_i, \quad = i,\ldots., n \text{ (equation 15)}$$

The operator equation (39) now takes the form

$$(\alpha \hat{U} + \hat{A})\, I + \beta_1 \psi^{m} + \beta_2 \psi^{o} = \hat{B} \qquad \ldots\ldots\ldots(40)$$

where $\hat{U}$ is the unit matrix.
The solution of the linear system (40) is obtained as follows.
Let $I^{(0)}, I^{(1)}, I^{(2)}$ be respectively, the solutions of the systems

$$(\alpha \hat{U} + \hat{A})\, I = \hat{B}, \qquad \ldots\ldots\ldots(41)$$

$$(\alpha \hat{U} + A)\, I = -\psi^{m} \qquad \ldots\ldots\ldots(42)$$

and

$$(\alpha \hat{U} + \hat{A})\, I = -\psi^{o} \qquad \ldots\ldots\ldots(43)$$

Then the solution to (40) is

$$I = I^{(0)} + \beta_1 I^{(1)} + \beta_2 I^{(2)} \qquad \ldots\ldots\ldots(44)$$

Accrodingly, the following procedure may be used for solving the system (40):

i) Assume a suitable value for the regularization parameter α.
ii) Solve the systems (41) - (43) to get $I^{(0)}$, $I^{(1)}$ and $I^{(2)}$.
iii) Substitute these values in (44) to get [redacted] of the Lagrangion multipliers β_1 and β_2.
iv) Apply the boundary conditions

$$P(U = U_m) = (I, \psi^{m}) = I \quad \text{(equation [redacted])}$$

and

$$P(U = U_o) = (I, \psi^{o}) = 0 \quad \text{(equation 31)}$$

to determine β_1 and β_2 and, consequently, the optimum excitation current I.

The choice of α depends on the allowable overall deviation of the synthesized pattern from the desired one, i.e. on $E = \|\hat{F}I - P_d\|^2$. However, the relation $E - \alpha$ is not linear, as seen from the numerical results in the next section. Therefore one may at first assume some trial values for α, solve the system (40) for these values, determine the error E in each case, and then apply an interpolation method to get the pertinent the value of α.

IV- NUMERICAL RESULTS

The technique described above has been applied to a number of example problems. Fig.(3) shows a sector beam pattern synthesized using 9- element array with halfwave interelement spacing

Fig. (4) shows the same pattern produced by a similar array, but having a quarter - wavelength interelement spacing. Different values of the regularization parameter α have been considered. The excitation currents in each case are indicated on the corresponding figure. For $\alpha = 0$, these currents are much larger than for other values, especially in the case of quarter-wavelength spacing. There, the currents in the innermost elements are excessively large and much larger than the currents in the elements near the ends of the array. Such unequal loading of the array elements diminishes its electrical reliability and is not recommended in any practical design. Also, the high degree of tapering in the current distribution over the operature results in an increased superdirectivity ratio and a corresponding decrease in antenna gain. The phenomenon is less troublesome in array having half wavelength interelement spacing. A similar remark has been made in [15] . It is seen from the results on Fig.(3) and Fig.(4) that, itroducing a small regularization parameter decreases the norm of the excitation current $\|I\|$, and, therefore, decreases the sensitivity of the array to construction errors. Also, the tapering of the aperture distribution is reduced, which improves the supordirectivity of the array. However, as seen from Fig. (4), this is done at the expense of increasing the deviation the(error E) of the synthesised pattern from the desired one. The curve on this figure shows that E increases monotonically with α. Therefore, one have to strike a compromise, accept more error E but reduce $\|I\|$ and consequently, the sensitivity to construction error to an acceptable level. This can be done by constructing a plot of $\|I\|$ and E as function of α as in Fig. (5), and then choosing an optimum value for α.

The value of E corresponding to $\alpha = 0$ is the minimum attainable error for an array of given length and number of elements. This value indicates the limiting case.

In the above examples the synthesized pattern is not constrained to have a definite main beam width. Therefore, β_2 was set equal to zero in equation (40).

If, however, the main beam width is specified rigorously, then we may follow the steps indicated in section III to synthesize the required array. This is the case of the example in Fig.(5). The object of this example is to design an array, having side lobe level as low as possible, and whose main beam has a width between nulls equal to 60° and is broader than that produced by a chebyshev array of the same length as the designed one. The desired pattern is synthesized using a center symmetric linear array of 9- elements as in Fig. (1). To facililate computations, the specified pattern has been expressed an alytically by:

$$P_d = 1 \qquad \|U\| \leq .5 \qquad \text{(equation 29)}$$

$$= 0 \qquad \|U\| > .5$$

$$U^o = .5 \qquad \text{(equation 31)}$$

As seen from Fig. (6), the above requirements ahave been acheived, but at the expense of a side lobe level higher than that of the chebyshev array.

V- CONCLUSION

The above analysis and numerical examples show that the variational method presented in this paper is a powerful and useful tool in the synthesis of linear arrays. The regularization technique involved in the method ensures the insensitivity of the synthesized array to design errors.

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APPENDIX

To prove that the set S is closed and convex

The set S is defined by

$$S = \left\{ I : \hat{F}_m I = 1, \hat{F}_o I = 0, \hat{F} I - P_d \leqslant E \right\} \qquad \ldots\ldots(1)$$

where I are points in the n-dimensional space of excitation current $\hat{F}$ is a linear continuous operator defined by:

$$\hat{F} I = (I, \psi) = \sum_m I_m \psi_m (u) \quad \text{(equation 2)}$$

$$\hat{F}_m I = (I, \psi(u_m)), \qquad \text{(equation (30))}$$
$$\hat{F}_o I = (I, \psi(u_o)), \qquad \text{(equation (31))}$$

and u_m, u_o - refer to the main beam and first zero of the pattern, respectively.

Consider a sequence of points of S: I^1, I^2, I^n,.... and let I^* be a limit point of this sequence. This implies that for any $\epsilon \geqslant 0$, there exists a positive integer m such that for all integers n m, the inequality $|I^* - I^n| \geqslant \epsilon$ is true. We shall prove that $I \in S$.

Let $\hat{F}_m I^* = a$, then

$$|a - 1| = |\hat{F}_m I^n - F_m I^*| = |F_m (I^* - I^n)|$$
$$\leqslant (\epsilon, \psi^n) \leqslant \|\psi^m\|$$

Taking the limit, we see that $\lim_n |a-1| = 0$ and hence a = 1, i.e.

$$\hat{F}_m I^* = 1 \qquad \ldots\ldots(2)$$

In the same way, it can be shown that

$$\hat{F}_o I^* = 0 \qquad \ldots\ldots(3)$$

To prove that I^* satisfies the third requirement in definition (1), we make use of the inequality

$$\left| \|Z_1\| - \|Z_2\| \right| \leqslant \|Z_1 - Z_2\|$$

valid for any two elements Z_1 and Z_2 of a Hilbert space.

Let $\|\hat{F}I^{*} - P_d\| = \Delta + \epsilon_1$ ......(4)

we have

$$\epsilon_1 \leqslant \left| \|\hat{F}I^{*} - P_d\| - \|\hat{F}I^{n} - P_d\| \right| \leqslant \|\hat{F}I^{*} - \hat{F}I^{n}\|$$

$$= \|\hat{F}(I^{*} - I^{n})\| \leqslant \epsilon \|\psi\|$$

Taking the limit, we see that;

$$\epsilon_1 \leqslant 0 \qquad ......(5)$$

Substituting into (4), we get

$$\|\hat{F}I^{*} - P_d\| \leqslant \Delta \qquad ......(6)$$

From (2), (3) and (6) we see that the limit point I^{*} of the given sequence satisfy all requirements of the definition (1) and hence belongs to the set S. This proves closeness of this set, To prove convexity of S, take any two points I_n and I_m of S and consider the point

$$I = t\, I_n + (1 - t)\, I_m \qquad 0 \leqslant t \leqslant 1 \qquad ......(7)$$

By definition, S is convex if I S. This fact follows from:

$$\hat{F}_m(t\, I_n + (1 - t)\, I_m) = t\, \hat{F}_m I_n + (1-t)\, \hat{F}_m I_m$$

$$= t + 1-t = 1 \qquad ......(8)$$

$$\hat{F}_o(t\, I_n + (1 - t)\, I_m) = t\, \hat{F}_m I_n + (1-t)\, \hat{F}_o I_m$$

$$= 0 \qquad ......(9)$$

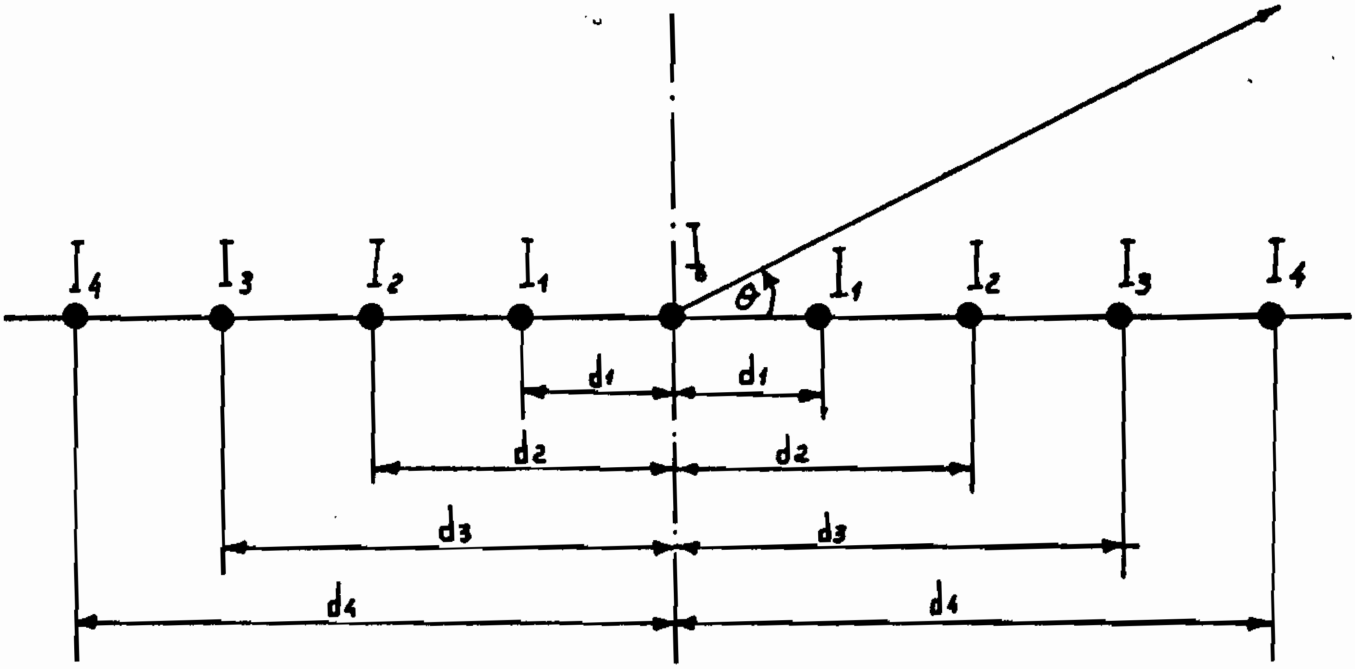
$$\|\hat{F}\,(t\, I_n + (1-t)\, I_m - P_d\|$$

$$= \|\hat{F}\, t\, I_n - t\, P_d + \hat{F}\,(1-t)\, I_m - (1 - t)\, P_d\|$$

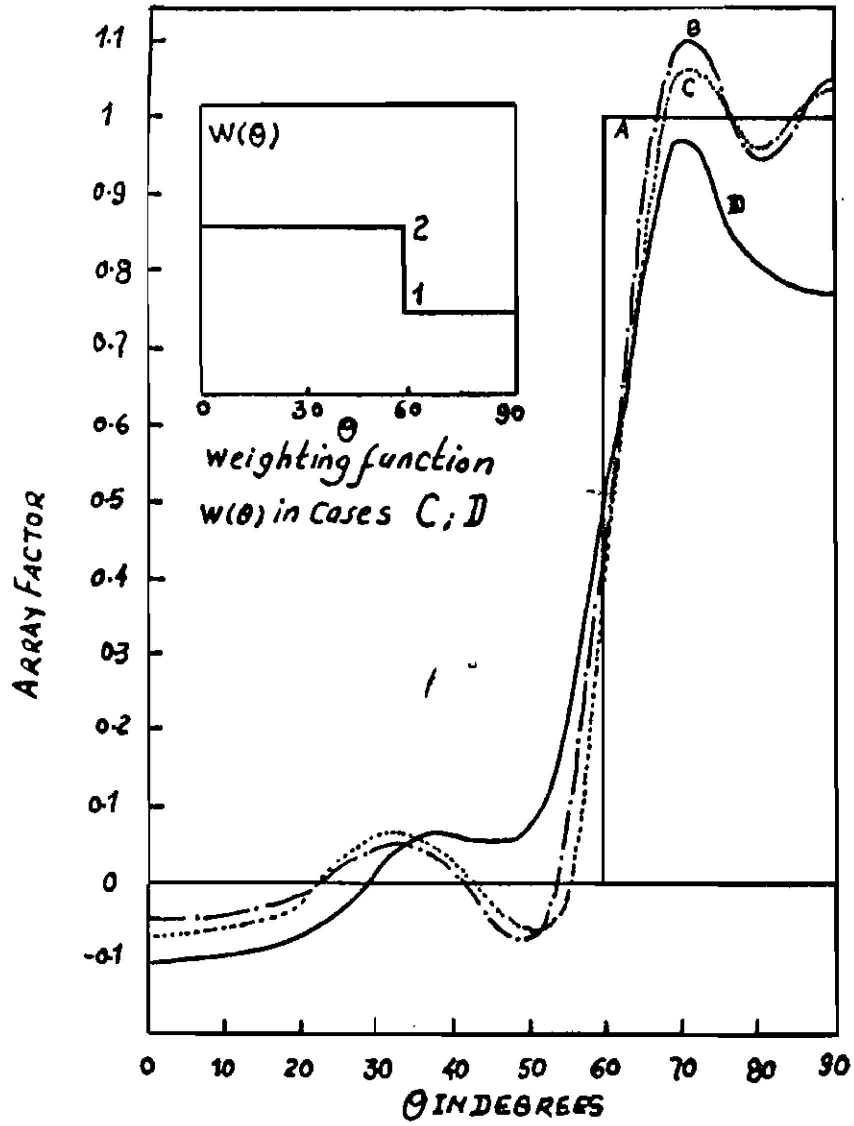
$$\leqslant t\, \|\hat{F}\, I_n - P_d\| + (1-t)\, \|\hat{F} I_m - P_d\|$$

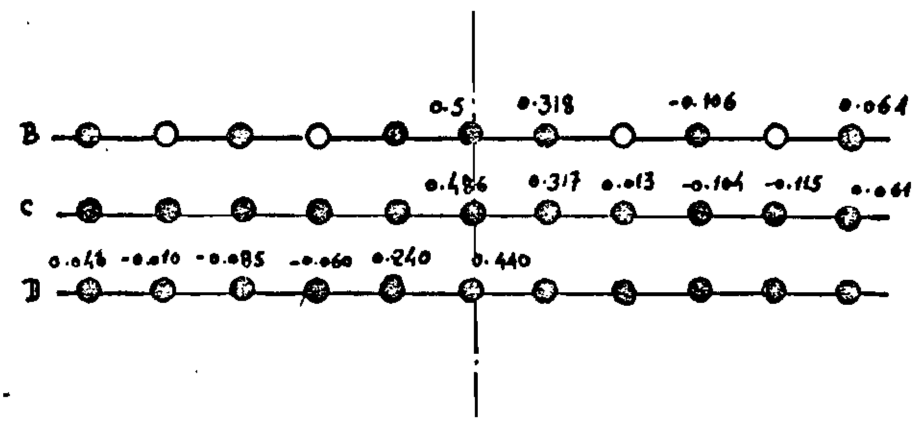
$$\leqslant t\Delta + (1-t)\Delta = \Delta \qquad ......(10)$$

From (7) through (10) we see that I satisfies all requirements of S and hence belongs to it. Thus, the S it convex.



Fig(1) Center-Symmetric Linear Array





(A) Desired pattern

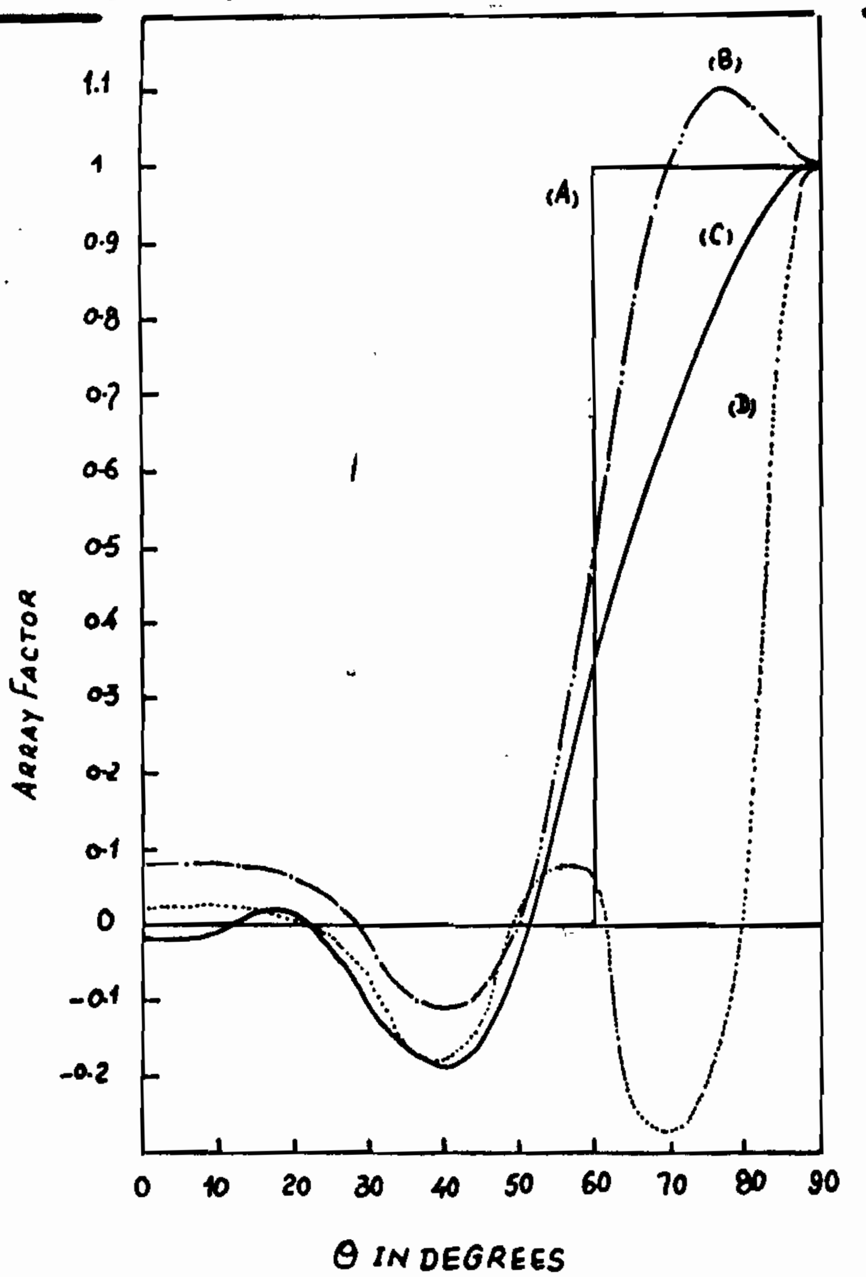
(B) L-2 Norm

(C) Weighted L-2 Norm

(D) Weighted Sobolev Norm

Fig (2) Synthesis of 11-element array using different Norms

(λ/2 spacing)



(A) Desired pattern (60° Sector beam)

(B) $\alpha = 0$

(C) $\alpha = 1$

(D) $\alpha = \infty$

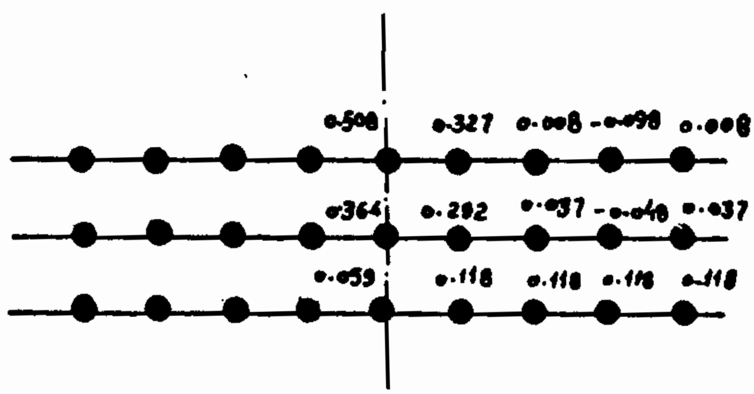


Fig (3) Radiation Pattern of 9-element Array for different α
($\lambda/2$ spacing)

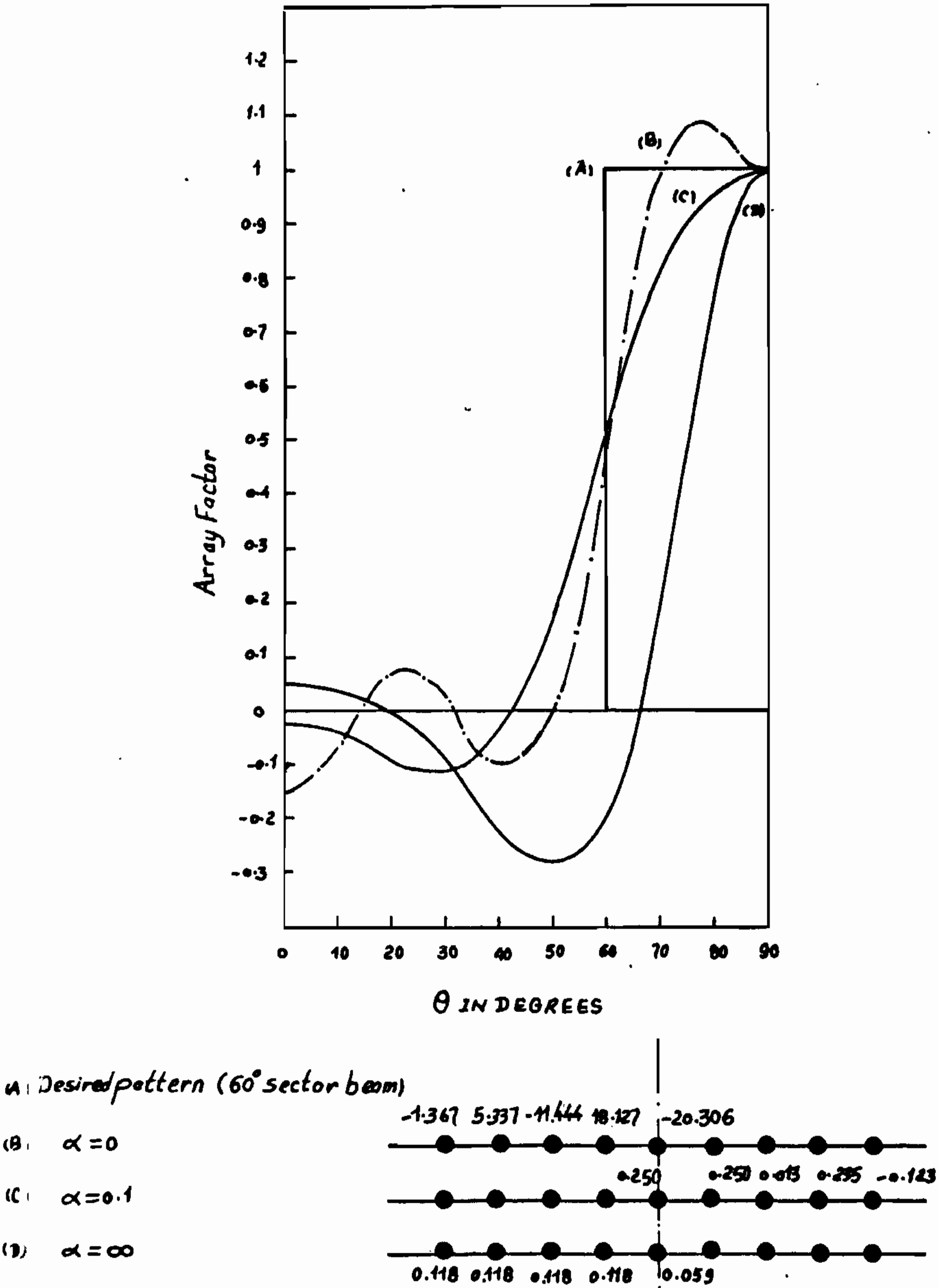
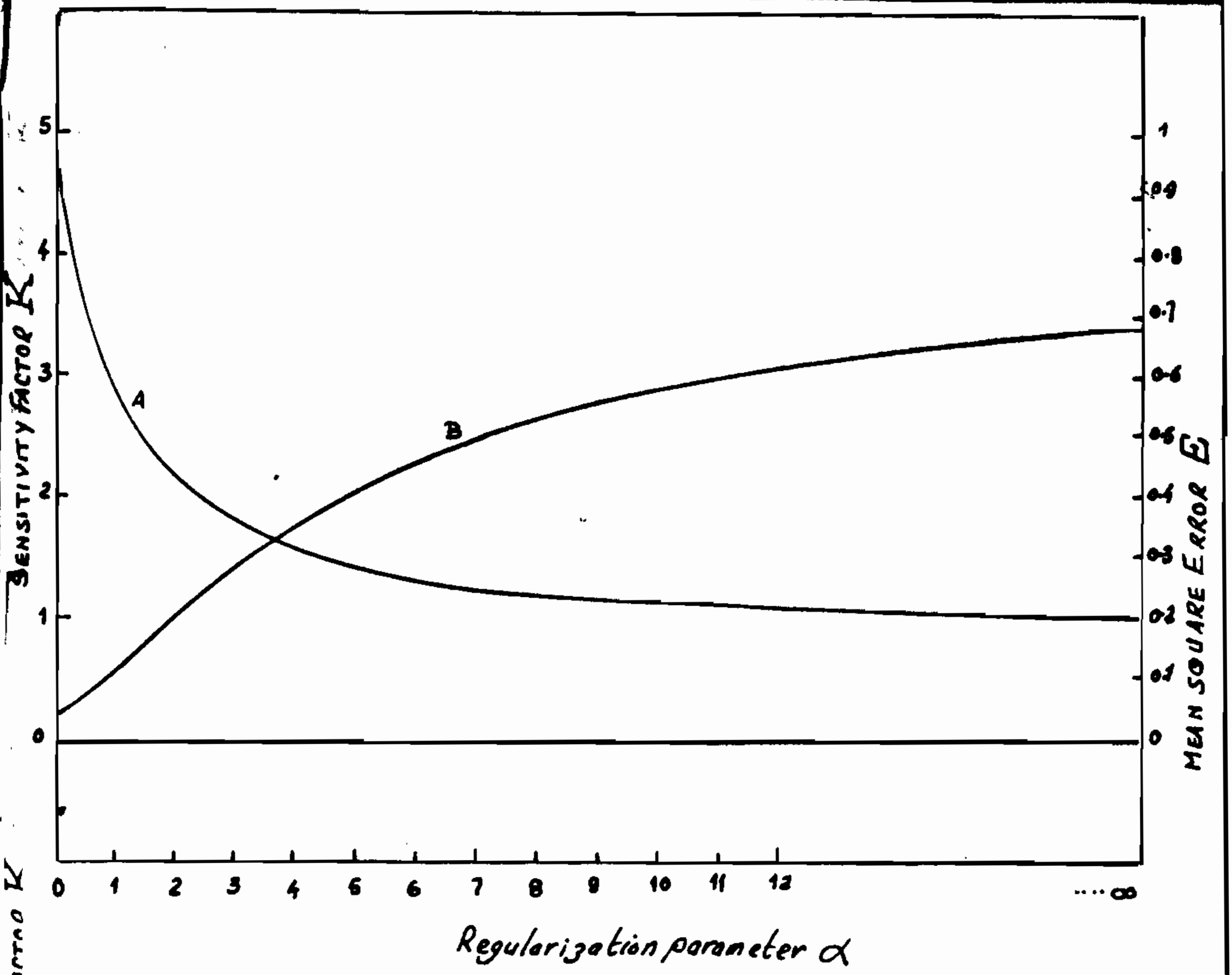


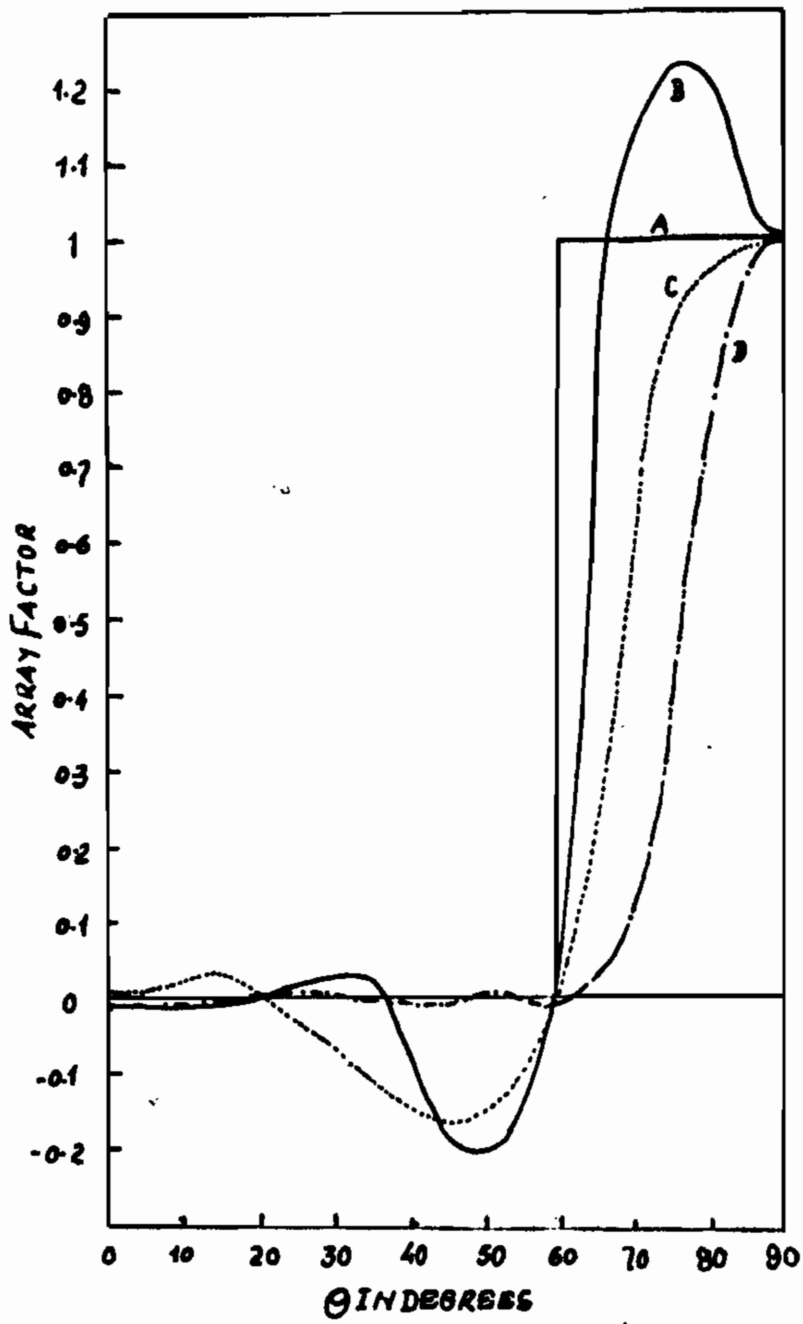
Fig (4) Synthesis of 9-element Array for different regularization parameter α
(λ/4 Spacing)



(A) K - α Relation

(B) E - α Relation

Fig(5) Dependance of sensitivity K and error E on parameter α
(9- element array; λ/2 spacing)



(A) Desired Pattern (60° sector beam)

(B) Variational Method ($\alpha = 0$)

(C) Variational Method ($\alpha = 1$)

(D) Chebyshev Method

Fig (6) Synthesis of 9-element array using different Methods

($\lambda/2$ spacing)