Mansoura University,
Faculty of Engineering,
Math. & Eng. Physics Dept.

Preparatory Students
Mathematics (1)

First Semester 2009/2010
Date: Monday 25/1/ 2010
Time allowed: 3 Hours

Answer the following questions [Full mark 130 points]

1. (a) Solve for x

(i) $\ln(x+2)=\ln(x-7)+\ln 4$, (ii) $\pi=2\sin^{-1}(x+4)$ [4+4 points]

(iii) $\dfrac{x^2-4x+3}{x+2}\geq 0$ [4 points]

(b) If $f(x)=\dfrac{1}{x}$ and $g(x)=\sin x$, compute $(g\circ f)(\dfrac{2}{\pi})$. [4 points]

(c) Evaluate each of the following limits

(i) $\lim\limits_{x\to 0}(3\csc x-\cot x)$, [4 points]

(ii) $\lim\limits_{x\to\infty}\dfrac{\sinh x}{3+\tan^{-1}x}$ [4 points]

(iii) $\lim\limits_{x\to 0^+}(1+\sin 3x)^{\csc x}$ [4 points]

(d) Find the domain, range and discuss the symmetry of $y=\tanh x$. Sketch the graph of this function and then prove that $\tanh(\ln x)=\dfrac{x^2-1}{x^2+1}$. [7 points]

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2. (a) Find $\dfrac{dy}{dx}$ for each of the following functions

(i) $y=\sec^{-1}(\cosh 5x)+\operatorname{csch}^{-1}(3x)$ [6 points]

(ii) $(\cos x)^y+\operatorname{sech} y=\cos^{-1}x+\log_3(x^4+1)$ [8 points]

(iii) $y=a(1-\cos t)$, $x=a(t-\sin t)$ [4 points]

(b) If $y=\sinh(m\sinh^{-1}x)$, prove that

$$\left(1+x^2\right)y''+xy'-m^2y=0,$$ [5 points]

hence, or otherwise, deduce that

$$\left(1+x^2\right)y^{(n+2)}+(2n+1)xy^{(n+1)}+(n^2-m^2)y^{(n)}=0.$$ [5 points]

(c) Find the Taylor's expansion of order 4 of the function $f(x)=\sin x$ about the point $x=\pi/3$ and hence evaluate $\cos 61^\circ$ from this expansion. [7 points]

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3. **(a)** Prove, by induction, that for each positive integer n, $5^n - 2^n$ is divisible by 3. [10 points]

(b) Given $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

(i) Plot in the Argand diagram each of the complex numbers

$$z_1, \quad \bar{z}_1, \quad -z_1, \quad \frac{1}{z_1}, \quad \frac{1}{\bar{z}_1}$$ [5 points]

(ii) Find $(z_1)^6$, $\frac{1}{(\bar{z}_1)^6}$, and $z_1 - \frac{1}{\bar{z}_1}$, [5 points]

(iii) Use de Moivre's theorem to solve the equation $z^3 + 2z_1 = 0$. [5 points]

(c) Using the binomial theorem, find the coefficient of x^n in the expansion $\left(\frac{2+x}{1-x}\right)^2$. [5 points]

(d) For nonsingular $n \times n$ matrices A and B, prove that $(AB)^{-1} = B^{-1}A^{-1}$. Hence show that $A^{-1}\left(A^{-1} + B^{-1}\right)^{-1} B^{-1} = (A+B)^{-1}$. [5 points]

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4. **(a)** If $x = -\frac{1}{2}$ is a root of the equation $8x^3 + 4x^2 + 2x + 1 = 0$, find the other two roots. [5 points]

(b) Given the polynomial $f(x) = 16x^5 + 16x^4 - x - 1$.

(i) Find the quotient and remainder when dividing $f(x)$ by $(2x^2 + x - 1)$. [5 points]

(ii) Solve the equation $f(x) = 0$. [5 points]

(iii) Find the equation whose roots are the reciprocals of twice of the roots of the equation $f(x) = 0$, and solve the new equation. [5 points]

(c) Prove that the product of orthogonal matrices is orthogonal. [5 points]

(d) Given the matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -1 & 2 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(i) Find $A^t - 2C + 3I$, C^{-1}, $CA\mathbf{b}$, $(C^{-1})^t C^t$, and $|2C|$. [5 points]

(ii) Solve the linear system of equations $A\mathbf{x} = \mathbf{b}$. [5 points]

With all best Wishes **Monday, 25/1/2010**