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Assessing the Lifetime Performance Index of Products with Bilal Distribution Under Progressively Type-II Censored Samples.

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Abstract: In industry, assessing the performance of products became very essential. Lifetime performance index C_L, is widely used in products assessing where L represents the lower specification limit. In many situations, the type II censored progressive data is required due to some constraints like budget, or there is a demand for repeat test. This paper is prepared to study the maximum likelihood estimator of the lifetime performance index, C_L based on the type II censored progressive data. Also a hypothesis test procedure is used to show whether the components lifetimes adhere to the desired levels in the conditions of pre-known L. Finally, real and simulated data are proposed as numerical examples to illustrate the theoretical study.

keywords: Bilal distribution; censored data; parameters estimation; lifetime performance index.

1.Introduction

Capability index of the process, "CIP" is a measure of the quality level in manufacturing process. There are three main types of capability index of the process, the first is "*the target the better type*" quality which is used when there is a desire to reach a particular target, the second is "*the larger the better type*" which includes the C_L index which is one of most recommended CIPs indices. Many papers discussed the lifetime performance index with normal and non-normal distributions, for more details see for example ([1], [3], [14], [15], [16], and [21]). Montgomery [17] studied the lifetime performance index and showed that C_L is one of the most recommended CIPs and used in many fields like hardness and lifetime. The last one is "*the smaller the better type*" which used in some fields like examination period of a product and the level of radiation.

There are many types of lifetime censoring scheme. One of the most commonly used is type-I and type-II censoring scheme which don't allow the experimenter to remove some components during the experiment. So, a very popular type called progressively type-II censoring with pre-planned censoring scheme arises which allow the experimenter to remove components and then reuses them or decreases the cost of the experiment. El-Sagheer et al. [9] used the ML, "maximum likelihood" and bayesian estimators for lifetime performance index with a known L, "lower specification limit" based on a progressive type-II censored data following power hazard function distribution. Many papers studied the type-II of censoring, for more details see([4], [11] and [22]). In addition, Hassan et al. [19] constructed the MLE of C_L based on the progressive Type II censored sample following the Burr type III distribution and utilized it to develop the hypothesis testing procedure in the condition of known L. Also, Shu-Fei et al. [20] developed the experimental design based on the testing procedure for the lifetime performance index of products following Weibull lifetime distribution under progressive type I censoring scheme and utilized the MLE to develop the hypothesis testing procedure.

On the other hand, many papers studied various types of lifetime censoring scheme, (see for example [13]). Balasooriya [6] and Wu and Ku [24] used the first failure censoring in which the experiment terminates when the first failure is monitored. In the same manner, Zhang and Gui [25] studied the lifetime performance index of a progressive type-II censored data following Pareto distribution and used MlE and BE methods for estimating the parameters.

Figure 1 shows the type-II of right censoring scheme sample such that $x_{k,n}, k = 1, \dots, m$ are the observed failure times, and $R_{k,n}, k = 1, \dots, m$ are the corresponding scheme of removed items with drawn from the test. It is noted that $x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{m,n}$. Let m be the number of failure items and R_i be the number of removed items due to the ith failure, $i = 1, 2, .., m-1$ and $R_m = n - m - \sum_{j=1}^{m-1} R_j$ is the removed items due to the mth failure which ends the experiment and $m, R = (R_1, R_2, \dots, R_m)$ are pre-assigned and $n - m = \sum_{i=1}^{m} R_i$.

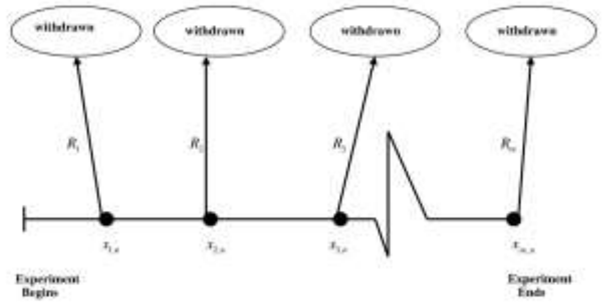

Fig1: Type-II of right censoring scheme sample.

The remaining of this paper is prepared as follows: in Section 2 some mainly properties of the lifetime index are proposed, Section 3 discusses the conforming rate and shows the relationship between the lifetime performance index and the conforming rate, Section 4 presents some methods of estimations including the maximum likelihood method and the bayesian method for a parameter following the Bilal distribution, Section 5 introduces a $(1-\alpha)100\,\%$ confidence interval for C_L and finally some numerical examples are introduced to illustrative the theoretical study in Section 6.

2. The Lifetime Performance Index

Lot of papers discussed the lifetime performance index with various distributions. In this paper, the C_L is discussed when the lifetime distribution is the Bilal distribution. Let X denote the lifetime distribution of items, then according to [2], the density and the cumulative functions of the random variable X following Bilal distributions are given as

$$f(x;\theta) = \frac{6}{\theta} e^{-\frac{2x}{\theta}} \left(1 - e^{-\frac{x}{\theta}}\right),$$

$$x \geq 0, \qquad \theta > 0, \qquad (1)$$

and

$$F(x;\theta) = 1 - e^{-\frac{2x}{\theta}} \left(3 - 2e^{-\frac{x}{\theta}}\right),$$

$$x \geq 0, \qquad \theta > 0. \qquad (2)$$

According to [16], the C_L is defined as

$$C_L = \frac{\mu - L}{\sigma}. \qquad (3)$$

From Equation (1), μ and σ can be obtained as follows

$$\mu = \frac{5\theta}{6}, \qquad \sigma = \frac{\sqrt{13}\theta}{6}, \theta > 0. \qquad (4)$$

Substituting from Equation (4) into Equation (3), the C_L is obtained as follows:

$$C_L = \frac{\frac{5\theta}{6} - L}{\frac{\sqrt{13}\theta}{6}} = \frac{5\theta - 6L}{\sqrt{13}\theta}. \qquad (5)$$

3. The Conforming Rate

The conforming item is the item with a lifetime that exceeds the lower limit of specification. Let X be the lifetime of an item, the item is conformed if $X \geq L$. Now let X follows Bilal distribution, then the conforming rate, P_r is defined as follows

$$P_r = P(X \geq L) = \int_L^\infty f(x;\theta) dx$$

$$= \int_L^\infty \frac{6}{\theta} e^{-\frac{2x}{\theta}} \left(1 - e^{-\frac{x}{\theta}}\right) dx = e^{-\frac{3L}{\theta}} (3e^{\frac{L}{\theta}} - 2)$$

$$= e^{-\frac{\sqrt{13}C_L - 5}{2}} \left(3e^{\frac{5 - \sqrt{13}C_L}{6}} - 2\right). \qquad (6)$$

It's noticed that P_r must achieve $0 \leq P_r \leq 1$ which implies $-\infty < C_L \leq 1.387$.

Table 1 shows some values of P_r for various values of C_L and it's noted that P_r increases when C_L increases.

Table 1: The conforming rate versus the lifetime performance index.

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.0000	-2	0.0468	0.2	0.4852	0.9	0.8392
-100	3.6×10^{-53}	-1	0.1433	0.4	0.5787	1	0.8888
-50	4.5×10^{-27}	-0.5	0.2440	0.5	0.6291	1.1	0.9328
-20	2.1×10^{-11}	-0.1	0.3654	0.6	0.6812	1.2	0.9686
-10	3.4×10^{-6}	0	0.4025	0.7	0.7343	1.3	0.9925
-5	0.0014	0.1	0.4424	0.8	0.7876	1.37	0.9997

Note: when $C_L \to 1.387, P_r \to 1$.

4. Parameter Estimation

In this section, maximum likelihood estimation "MLE" and bayesian estimation "BE" methods are used to estimate the θ parameter of C_L.

4.1 Maximum likelihood estimation method

In many experiments of the lifetime tests, it is wanted to remove some products of the experiment due to some restrictions such as lack of funds, lack of materials and so on. In this paper, it is considered the case of type II progressive censoring scheme.

Let X represent the lifetime of the products and let X follow Bilal distribution given in [2]. Let n presents the number of whole items placed on the test. Further, let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{m,n}$ be the corresponding censoring sample of type II with corresponding censoring scheme, $R = (R_1, R_2, \ldots, R_m)$. According to [4], the joint density function of $X_{1,n}, X_{2,n}, \ldots, X_{m,n}$ is given by

$$L(\theta) = c\prod_{i=1}^{n} f_X(x_i, n)\big(1 - F_X(x_i, n)\big)^{R_i}, \quad (7)$$

where $c = n(n - R_1 - 1)\ldots(n - R_1 - R_2 - -R_{m-1} - m + 1)$.

Substituting from Equations (1) and (2) into Equation (7) we get

$$L(\theta) = c\prod_{i=1}^{n}\frac{6}{\theta}e^{-\frac{2x_i}{\theta}}\left(1 - e^{-\frac{x_i}{\theta}}\right) \times \left(e^{-\frac{2x_i}{\theta}}(3 - 2e^{-\frac{x_i}{\theta}})\right)^{R_i}, \quad (8)$$

and the natural logarithm, l is

$$l = \ln(c) + m ln(6) + m ln\left(\frac{1}{\theta}\right) - \frac{2}{\theta}\sum_{i=1}^{m} x_i - \frac{2}{\theta}\sum_{i=1}^{m} x_i R_i + \sum_{i=1}^{m} ln(1 - e^{-\frac{x_i}{\theta}}) + \sum_{i=1}^{m} R_i \ln\left(3 - 2e^{-\frac{x_i}{\theta}}\right).$$

Now, let $\eta = \frac{1}{\theta}$, then l can be written as

$$l = \ln(c) + m ln(6) + m ln(\eta) - 2\eta\sum_{i=1}^{m} x_i - 2\eta\sum_{i=1}^{m} x_i R_i + \sum_{i=1}^{m} ln(1 - e^{-\eta x_i}) + \sum_{i=1}^{m} R_i \ln(3 - 2e^{-\eta x_i}).$$

The first derivative of l with respect to η can be given as

$$\frac{\partial l}{\partial \eta} = \frac{m}{\eta} - \sum_{i=1}^{m}\left(2x_i + 2x_i R_i - \frac{x_i e^{-\eta x_i}}{1 - e^{-\eta x_i}} - \frac{2x_i R_i e^{-\eta x_i}}{3 - 2e^{-\eta x_i}}\right) = 0,$$

then,

$$\frac{m}{\eta} = \sum_{i=1}^{m}\left(2x_i + 2x_i R_i - \frac{x_i e^{-\eta x_i}}{1 - e^{-\eta x_i}} - \frac{2x_i R_i e^{-\eta x_i}}{3 - 2e^{-\eta x_i}}\right).$$

It is obvious that the left hand side is a decreasing function of η. Now define the function $h(x;\eta)$ such that

$$h(x;\eta) = \sum_{i=1}^{m}\left(2x_i + 2x_i R_i - \frac{x_i e^{-\eta x_i}}{1 - e^{-\eta x_i}} - \frac{2x_i R_i e^{-\eta x_i}}{3 - 2e^{-\eta x_i}}\right).$$

It is noted that $lim_{\eta\to\infty} h(x;\eta) = 2n\bar{x}\,(R_i + 1), lim_{\eta\to 0}\, h(x;\eta) = -\infty$, such that $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$. The first derivative of the function $h(x;\eta)$ is given by

$$\frac{\partial h(x;\eta)}{\partial \eta} = \sum_{i=1}^{m}\left(\frac{x_i^2 e^{-\eta x_i}}{(1 - e^{-\eta x_i})^2} + \frac{6x_i^2 R_i e^{-\eta x_i}}{(3 - 2e^{-\eta x_i})^2} > 0,\right.$$

then, $h(x;\eta)$ is an increasing function of η and intersects with $\frac{m}{\eta}$ at η^*. Figure 2 shows an asymptotic figure of the functions $h(x;\eta)$ and $\frac{m}{\eta}$.

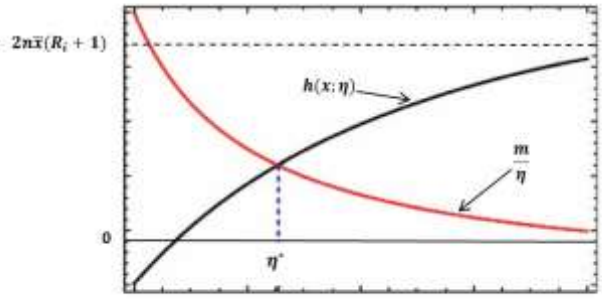

Fig 2: Asymptotic figure of the functions $h(x;\eta)$ and $\frac{m}{\eta}$.

Therefore, the ML-estimate of $\hat{\theta}$ is obtained as

$$\hat{\theta} = \frac{1}{m}\sum_{i=1}^{m}\left(2x_i + 2x_i R_i - \frac{x_i e^{-\frac{x_i}{\hat{\theta}}}}{1 - e^{-\frac{x_i}{\hat{\theta}}}} - \frac{2x_i R_i e^{-\frac{x_i}{\hat{\theta}}}}{3 - 2e^{-\frac{x_i}{\hat{\theta}}}}\right). \quad (9)$$

By using the invariance of MLE (see [20]), C_L given in Equation (3) can be rewritten as

$$C_{L_{\hat{\theta}}} = \frac{5\hat{\theta} - 6L}{\sqrt{13}\hat{\theta}}, \quad (10)$$

where $\hat{\theta}$ is the MLE_{θ} given in Equation (9).

Further, the asymptotic normal distribution of the maximum likelihood estimate is given as follows

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{m}{\theta^2} - \frac{4(\sum_{i=1}^{m} x_i R_i + \sum_{i=1}^{m} x_i)}{\theta^2} + \sum_{i=1}^{m}\left(\frac{x_i e^{-\frac{x_i}{\theta}}\left(2\theta\left(1 - e^{-\frac{x_i}{\theta}}\right) - x_i e^{-\frac{x_i}{\theta}}\right)}{\theta^4\left(1 - e^{-\frac{x_i}{\theta}}\right)^2} - \frac{x_i^2 e^{-\frac{x_i}{\theta}}}{\theta^4\left(1 - e^{-\frac{x_i}{\theta}}\right)^2}\right) +$$

$$\sum_{i=1}^{m}\left(\frac{2x_iR_ie^{-\frac{x_i}{\hat{\theta}}}\left(2\theta\left(3-2e^{-\frac{x_i}{\hat{\theta}}}\right)-2x_ie^{-\frac{x_i}{\hat{\theta}}}\right)}{\theta^4\left(3-2e^{-\frac{x_i}{\hat{\theta}}}\right)^2}-\frac{2x_i^2R_ie^{-\frac{x_i}{\hat{\theta}}}}{\theta^4\left(3-2e^{-\frac{x_i}{\hat{\theta}}}\right)^2}\right).(11)$$

Under some regularity conditions and for large samples, the asymptotic normal distribution for the MLE of θ can be given as (see [21])

$$\hat{\theta}\sim N(\theta, I(\theta)^{-1}, \qquad (12)$$

where $I(\theta)$ represents the Fisher information matrix of θ and is defined as negative of the expectation of Equation (11). However, it is very difficult to obtain this expectation in a closed form, then the approximate information matrix $I_0(\hat{\theta})$ is used and is given as

$$I_0(\hat{\theta})=\left[-\frac{\partial^2 l}{\partial\theta^2}\right]_{\hat{\theta}}=[v(\theta)]_{\hat{\theta}}. \qquad (13)$$

Also, the variance-covariance matrix is defined as follows

$$I_0(\hat{\theta})^{-1}=[v(\theta)]^{-1}_{\theta=\hat{\theta}}=[var(\hat{\theta})]_{\theta=\hat{\theta}}. \qquad (14)$$

Let $C_L\equiv C(\theta)$, and then according to [8] and using the multivariate delta method, the asymptotic normal distribution of $C(\hat{\theta})$ is

$$C_{L_{\hat{\theta}}}\equiv C(\hat{\theta})\sim N(C_L,\Psi_\theta). \qquad (15)$$

Also, the asymptotic variance-covariance matrix $\Psi_{\hat{\theta}}$ of $C(\theta)$ to estimate Ψ_θ is given as follows

$$\Psi_{\hat{\theta}}=\left[\frac{\partial C(\theta)}{\partial\theta}\right]I_0(\theta)^{-1}\left[\frac{\partial C(\theta)}{\partial\theta}\right]_{\hat{\theta}}. \qquad (16)$$

4.2 Bayesian estimation method

In this subsection, bayesian estimation (BE) approach is used to estimate the parameter θ assumed to follow gamma prior distribution with parameters a and b.

Gamma prior density function can be written as

$$g(u;a,b)=\frac{b^a}{\Gamma(a)}u^{a-1}e^{-ub}, u,a,b>0, \quad (17)$$

and the joint prior density of θ is given by

$$g(\theta)=\prod_{i=1}^{n}g(\theta_i)\propto\theta^{a-1}e^{-\theta b}. \qquad (18)$$

The joint posterior distribution function according to BE procedure is given by

$$g(\theta|\underline{x})=\frac{g(\theta)L(\underline{x})}{\int g(\theta)L(\underline{x})\,d\theta}\propto g(\theta)L(\underline{x}). \quad (19)$$

Substituting from Equations (8) and (18) into Equation (19) we get

$$\hat{\theta}=E(\theta)=\int_{\theta}\theta\, g(\theta|\underline{x})\,d\theta. \qquad (21)$$

The expectation given in Equation (21) has no closed form so Marcov chain Monte Carlo "MCMC" procedure is used.

MCMC procedure is used to summarize the posterior distribution numerically with no need to calculate the normalized constant, see [7]. The metropolis-hasting "M-H" algorithm is given below to illustrate the required steps of sampling:

1. Set the initial guess of $\theta^{(0)}$.
2. Let $j=1$.
3. Generate the proposed $\theta^{(*)}$ from $(\theta^{(j-1)}, var\,(\theta))$.
4. Match the acceptance function
$$\rho=min[1,\frac{g(\theta^{(*)}|\underline{x})}{g(\theta^{(j-1)}|\underline{x})}].$$
5. Generate u_1 from the uniform distribution.
6. If $u_1<\rho$, accept $\theta^{(*)}$ and then $\theta^{(j)}=\theta^{(*)}$, else $\theta^{(j)}=\theta^{(j-1)}$.
7. Set $j=j+1$.
8. Continue repeating steps (3) to (7) N times to obtain $\theta^{(i)}, i=1,2,\ldots,N$.

Removing first M trials of selection of initial values guaranteing the convergence we obtain

$$\hat{\theta}=\frac{1}{N-M}\sum_{j=M+1}^{N}\theta^{(j)}. \qquad (22)$$

$C_{L_{\hat{\theta}}}$ can be obtained by substituting from Equation (22) in to Equation (10), where $\hat{\theta}$ is the BE_θ given in Equation (21) or (22).

5. Testing Hypotheses Procedure for the Lifetime Performance Index

One-sided hypothesis testing procedure is constructed in this section in addition to one-sided confidence interval. They are constructed to show if the life time performance index, C_L adheres to the desired level, L. Let C^* denote the target or the desired value, thus the null hypothesis H_0 versus the alternative hypothesis H_1 are

$$H_0: C_L\leq C^*,$$
$$H_1: C_L> C^*.$$

According to [8] and [22], C_L has asymptotic normal distribution for large samples and $C_{L_{\hat{\theta}}}$ is the test statistic value and θ is estimated with either "MLE" or "BE" methods. The critical (rejection) region is then $C_{L_{\hat{\theta}}}$ where $C_{L_{\hat{\theta}}} > C_0$ and C_0 represents the critical value at a specific significance level α and is given as

$$P\left(\frac{C_{L_{\hat{\theta}}} - C_L}{\sqrt{\Psi_{\hat{\theta}}}} = \frac{C_0 - C^*}{\sqrt{\Psi_{\hat{\theta}}}}\right) = 1 - \alpha,$$

where $\frac{C_{L_{\hat{\theta}}} - C_L}{\sqrt{\Psi_{\hat{\theta}}}} \sim N(0,1)$ and thus $\frac{C_0 - C^*}{\sqrt{\Psi_{\hat{\theta}}}} = z_\alpha$.

The critical value, C_0 can be written as

$$C_0 = C^* + z_\alpha \sqrt{\Psi_{\hat{\theta}}}. \quad (23)$$

Also, the $(1-\alpha)100\ \%$ one-sided confidence interval of C_L can be obtained by

$$C_L \geq C_{L_{\hat{\theta}}} - z_\alpha \sqrt{\Psi_{\hat{\theta}}},$$

and $\underline{LB}$, the lower bound confidence interval of C_L is then

$$\underline{LB} = C_{L_{\hat{\theta}}} - z_\alpha \sqrt{\Psi_{\hat{\theta}}}. \quad (24)$$

The following steps summarize the testing procedure:

Step 1: Fit the data to Bilal distribution.

Step 2: Determine L, the lower limit of the lifetime of the products and the target value, C^*.

Step 3: Constructing the null and the alternative hypothesizes as $H_0: C_L \leq C^*$, $H_1: C_L > C^*$ respectively.

Step 4: Specify α, the level of significance.

Step 5: Calculate the test statistic value, $C_{L_{\hat{\theta}}}$ given in Equations (10, 23) and the $(1-\alpha)100\ \%$ lower bound of the confidence interval, $\underline{LB}$ given in Equation (24).

Step 6: Take the decision as follows:

If $C^* \notin [\underline{LB}, \infty)$, then we reject H_0, otherwise we accept H_0.

The rejection of H_0 means that the performance index of products lifetimes meets the desired level.

6. Numerical Examples

Some numerical examples based on real data and simulated data under Bilal distribution are proposed in this section to illustrate the theoretical procedure. MLE and BE methods are used to estimate the θ parameter using MCMC algorithm which performed 1000 times.

Example 6.1. The data below are survival times for a sample of twenty male rats that were exposed to a high radiation level (see [10]). The times are measured by weeks.

152, 152, 115, 109, 137, 88, 94, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83, 69.

Step 1: Fitting the data.

The data is fitted to Bilal distribution using the Mathematica program and the P-value is $0.17 > 0.05$ and then it follows Bilal distribution. Table 2 shows a sample of twelve random removals male rats from the previous data with a corresponding scheme. The MLE and BE estimated of θ and $\hat{\theta}$ are 175.97 and 173.12 respectively.

Table 2: Progressive censored survival times for twelve male rats exposed to a high radiation level.

i	**1**	**2**	**3**	**4**	**5**	**6**
$X_{i:m:n}$	152	115	109	137	160	125
R_i	0	1	1	1	0	0
i	7	8	9	10	11	12
$X_{i:m:n}$	40	128	123	136	62	153
R_i	1	1	1	0	1	1

Step 2: Let L of the survival times of the rats is 59.56, ie, if the survival time of a rat that was exposed to a high level of radiation exceeds 59.56 weeks then it is said to be a conforming rat. Then, the conforming rats rate, P_r must exceed 80 % and referring to Table 1, the corresponding C_L is 0.9. As a result, the value of the performance index is set as $C^* = 0.9$.

Step 3: The testing null and alternative hypothesizes must be: $H_0: C_L \leq C^*$, $H_1: C_L > C^*$ respectively.

Step 4: Let the level of significance, $\alpha = 0.05$.

Step 5: The lower bound of the 95 % confidence interval, $\underline{LB}$ given in Equation (24) using both MLE and BE methods is 0.76 and 0.753 respectively.

Step 6: Since $C^* = 0.9 \in [0.76, \infty)$ ($C^* = 0.9 \in [0.753, \infty)$), the null hypothesis H_0 can't be rejected, i.e, the rat lifetime performance index doesn't have the required level.

Example 6.2. The data below are failure times of fifteen electronic items in an accelerated life test (see [12]). The times are measured in minutes

1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23.0, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2.

Step 1: Fitting the data.

The data is fitted to Bilal distribution using the Mathematica program and the P-value is $0.98 > 0.05$ and then it follows Bilal distribution. Table 3 shows a sample of ten random removals electronic items of the previous data with a corresponding scheme. The MLE and BE estimated of θ and $\hat{\theta}$ are 39.57 and 38.71 respectively.

Table 3: Progressive censored failure times of ten electronic items in an accelerated life test.

i	1	2	3	4	5	6	7	8	9	10
$X_{i:m:n}$	1.4	5.1	10.8	12.1	18.5	22.2	30.6	37.3	59.8	66.2
R_i	0	0	1	0	0	2	1	0	1	0

Step 2: Let the lower limit of the lifetime of the products is 8.62, ie, if the failure time of an electronic item exceeds 8.62 minutes then it said to be a conforming item. Then, the conforming items rate, P_r must exceed 90 % and referring to Table 1, the corresponding C_L is 1.1. Thus, the value of the performance index is $C^* = 1.1$.

Step 3: The null and alternative hypothesizes must be: $H_0: C_L \leq C^*, H_1: C_L > C^*$ respectively.

Step 4: Let the level of significance, $\alpha = 0.01$.

Step 5: The lower bound of the 99 % confidence interval, $\underline{LB}$ given in Equation (24) using both MLE and BE methods is 1.3 and 1.28 respectively.

Step 6: Since $C^* = 1.1 \notin [1.3, \infty)$ ($C^* = 1.1 \notin [1.28, \infty)$), the null hypothesis H_0 is rejected, i.e, the performance index of items lifetimes has the desired level.

Example 6.3. A progressively type II censored data were generated from Bilal distribution for different n and m values and different values of parameters a and b. The mean square error (MSE) and coverage probabilities (CPs) are calculated to compare the estimators. The generated samples were due to Balakrishnan and Sandhu algorithm (see [5]). The following schemes were considered (see [18])

Scheme 1: $R_1 = n - m, R_i = 0$ for $i \neq 1$.

Scheme 2: $R_{\frac{m+1}{2}} = n - m, R_i = 0$ for $i \neq \frac{m+1}{2}$ and m is odd,

$R_{\frac{m}{2}} = n - m, R_i = 0$ for $i \neq \frac{m}{2}$ and m is even.

Scheme 3: $R_m = n - m, R_i = 0$ for $i \neq m$.

Tables 4 and 5 show the generated sample with a corresponding censoring schemes for different values of a and b.

Table 4: MSE of MLE and BE estimates methods with $a = b = 1$.

(n, m)	*scheme*	MLE_{C_L}	BE_{C_L}
$(30, 20)$	1	0.000095	3×10^{-6}
	2	0.000096	2×10^{-6}
	3	0.000095	3×10^{-6}
$(40, 20)$	1	0.000095	3×10^{-6}
	2	0.000095	3×10^{-6}
	3	0.000095	3×10^{-6}
$(40, 30)$	1	0.000095	2×10^{-6}
	2	0.000094	2×10^{-6}
	3	0.000093	2×10^{-6}
$(60, 40)$	1	0.000093	1×10^{-6}
	2	0.000093	1×10^{-6}
	3	0.000093	1×10^{-6}

Table 5: MSE of MLE and BE estimates methods with $a = b = 2$.

$(\boldsymbol{n}, \boldsymbol{m})$	***scheme***	$\boldsymbol{MLE_{C_L}}$	$\boldsymbol{BE_{C_L}}$
$(30, 20)$	1	0.000097	3×10^{-6}
	2	0.000096	3×10^{-6}
	3	0.000095	3×10^{-6}
$(40, 20)$	1	0.000095	3×10^{-6}
	2	0.000095	3×10^{-6}
	3	0.000095	3×10^{-6}
$(40, 30)$	1	0.000095	2×10^{-6}
	2	0.000094	2×10^{-6}
	3	0.000093	2×10^{-6}
$(60, 40)$	1	0.000093	1×10^{-6}
	2	0.000093	1×10^{-6}
	3	0.000093	1×10^{-6}

Tables 6 and 7 show the CPs of 95 % credible and confidence intervals for different values of a and b.

Table 6: CPs of 95 % credible and confidence intervals for $a = b = 1$.

(n, m)	$scheme$	MLE_{C_L}	BE_{C_L}
$(30, 20)$	1	0.9633	0.9627
	2	0.9533	0.9527
	3	0.9534	0.9626
$(40, 20)$	1	0.9533	0.9626
	2	0.9533	0.9626
	3	0.9533	0.9627
$(40, 30)$	1	0.9534	0.9628
	2	0.9534	0.9627
	3	0.9535	0.9628
$(60, 40)$	1	0.9535	0.9628
	2	0.9535	0.9629
	3	0.9535	0.9628

Table 7: CPs of 95 % credible and confidence intervals for $a = b = 2$.

(n, m)	$scheme$	MLE_{C_L}	BE_{C_L}
$(30, 20)$	1	0.9533	0.9626
	2	0.9533	0.9526
	3	0.9534	0.9626
$(40, 20)$	1	0.9533	0.9626
	2	0.9533	0.9626
	3	0.9533	0.9626
$(40, 30)$	1	0.9534	0.9627
	2	0.9534	0.9627
	3	0.9535	0.9627
$(60, 40)$	1	0.9535	0.9629
	2	0.9535	0.9628
	3	0.9535	0.9628

Referring to Tables (4, 5, 6 and 7) we notice the following

- As m, n increase, the MSE of MLE and BE estimates methods decrease, also the censoring scheme III, $R = (0, 0 \ldots, n - m)$ usually provides the smallest MSE for all estimators and as a sequence is the most efficient for all choices.
- The Bayes estimates method is generally smaller than their corresponding MLEs and become smaller when n and m become larger.
- The coverage probabilities (CPs) of one-sided credible interval for lifetime performance index C_L close to the desired level of 0.95 and so as the one-sided confidence interval (CI).

7. Conclusions

Assessing product performance is very important in various fields, especially in the manufacturing fields. The main objective of this evaluation is to meet the required level of quality among customers given some certain constraints such as budget, location, cost and other restrictions that cause obstacles when applying. Therefore, this paper studied the statistical inference of the lifetime performance index, one of the most capability indices, under type II right censored progressive samples having Bilal distribution. Two methods of parameter estimation including "maximum likelihood estimation method" and "bayesian estimation method" are used. The hypothesis testing procedure provided if the product lifetime meets the required level. Finally, some numerical examples are proposed on real data sets with two different schemes and simulated data with three different schemes to illustrate the theoretical study.

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