Sci. J. Fac., Sci., Minufiya Univ., Vol. XVIII (2004 : 123 - 139)

COMPUTER SIMULATION OF HEAT TRANAFER USING MONTE CARLO TECHNIQUE

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Received : 1/5/2003

Abstract

This work considers the application of the computer simulation technique as a problem solver tool. The heat distribution has been solved using the Monte Carlo simulation approach. We applied two different types of the heat distribution problem. The first one is the one-dimensional space, "heating metal bar" problem. The second problem is the two-dimensional space, "house heating by the sun". Both simulation and numerical results have been compared. The finite difference technique has been used in the numerical solution. The results show a good agreement for both approaches.

Introduction

Today, simulation analysis is a powerful problem-solving technique. Its origin lies in statistical sampling theory and analysis of complex probabilistic physical systems. The common thread in both of these is the use of random numbers and random sampling to approximate an outcome or solution. Studying the behavior of the system by this method becomes a necessity in several situations where we have either no other alternatives or the alternatives available are, not efficient, for more details see Law [1], Neelamkavil [2], and Alan[3].

H. Gould [4], and Hockney [5], used random walks to characterize the motion of a dust partical in a glass of water. The original statement of a random walk was formulated in the context of a "drunken sailor." If a drunkard begin at a lamp post and takes N steps of equal length in random direction, how far will the drunkard walker grows linearly with time? This result and its relation to diffusion leads to many applications that might seem to be unrelated to random walks.

In this work we implement a random walks idea to characterize two different problems. The first problem we used is the well-known "heated metal bar" model. This model characterized the heated distribution metal bar implementing random walks approach to visit right or locations successively. At time zero, the bar temperature distribution follows the initial condition imposed. When the metal bar is heated a new level of temperature maintained throughout the duration of the simulation, for more details see Hoover [6].

The second example represents a house heated by sun from the top and sides such that the temperature at the bottom remains fixed and equals to zero. The problem is how to compute the temperature distribution through the house, using Monte Carlo simulation, for more details see Texler et al, [7].

The results of the Monte Carlo simulation for both examples have been compared with numerical solutions using the finite difference method. Good agreements have been found for the both examples.

Heated Metal Bar Model

We first consider one-dimensional model, "heated metal bar". The heated metal bar is fully covered by perfect insulation except at one point, the mid point. The two ends of the metal bar are kept to be with zero Celsius. The objective is how to compute the temperature distribution over the whole bar during the simulation time? Initially, the metal bar temperature is uniformly zero on a normalized scale. A step increase in temperature is applied at the un-insulation mid point, and the new level is maintained throughout the duration of the simulation. Theoretically, the temperature distribution obeys the initial condition imposed. For comparison sake, we used the same initial condition imposed for both simulation and numerical approaches.

1- Simulation Approach

We intend to adopt the random walk technique to simulate the temperature distribution of the metal bar. The metal bar is heated continuously at the un-insulation mid point of the bar. In order to apply the random walk technique in our simulation study, we divide the bar length, **L**, into **n** equal intervals with space **h** = **L/n**. The walker begins to move randomly to the right or left of the un-insulation, mid bar, point. At each interval of time the walker has a probability **p** of a step to right and a probability **q (= 1-p)** of a step to the left. The direction of each step is independent of the preceding one.

For simplicity, we assume that the probabilities **p** and **q** are equal, i.e., **p = q = 1/2.** The random walk movement to right or to left of the current grid point is dependent on the value of **p** and **q**. Determination of the movement orientation is required to generate a uniformly distributed independent random number **rand** (0, 1). The random walk moves according to the following:

if **rand** > ½ then

visit right grid point

else

visit left grid point

When the random walk is performed repeatedly, **nt**, times, an estimate of the probability of the visiting each grid point, **P**, can be obtained. The probability **P** is computed from the ratio of the number of grid point visits, **N**, to the total number of trials, **nt**, i.e., **P = N/nt**. This technique is considered as a Monte Carlo simulation, since we use the random numbers to approximate the outcome.

The temperature $\mathbf{u_i}$ at each grid point will be computed according to the accumulated walker visits and the initial temperature value at that grid point. The initial temperature at each grid point, $U(x_i)$, is evaluated from the initial condition imposed. In our study, we introduced the initial condition as:

$$\mathbf{U\ (x_i) = 4\ x_i\ (1- x_i), \quad for \quad i = 1, \ldots, n-1} \qquad \text{-------} \quad (1)$$

Where

$$\mathbf{x_i = i*h}$$

In program we used the array element **p_cum (i)** to accumulate the number of grid point's visit after **nt** trial. The probability of visiting the $\mathbf{i^{th}}$ point is

$$\mathbf{P_i = p\_cum\ (i)\ /\ nt\ , \quad for\ \ i = 1, 2, \ldots, n} \qquad \text{-------} \quad (2)$$

Now, the temperature at each grid point can be computed as:

$$\mathbf{U_i = [\ P_i * U\ (x_i)\ ]\ /\ [2\ (n-1)]\ , \quad for\ i = 1, 2, \ldots, n} \qquad \text{-------} \quad \mathbf{(3)}$$

In order to increase the computational accuracy, we perform "**k**" of those random moves at each grid point, x_i, then accumulated the number of visit and compute the probability of each grid point. So, the average of the tempreature $\mathbf{u_i}$ at $\mathbf{x_i}$ is obtained.

2- Numerical Approach

Problem involving time, t, as one independent variable leads usually to parabolic equations. The simplest parabolic equation derived from the theory of heat conduction is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{--------} \quad (4)$$

Its solution gives the temperature U at a distance x units of length from mid point of a thermally insulated bar after t seconds of heat conduction. In such a problem the temperatures at the ends of a bar of length L are often known for all time. In other words, the boundary conditions are known. It is also usual for the temperature distribution along the bar be known at some particular instant. This instant is usually taken as zero time and the temperature distribution is called the initial condition. The solution gives U for values of x between 0 and L and values of t from zero to infinity.
The boundary condition is:

$$U(L) = U(0) = 0.0 \qquad \text{--------} \quad (5)$$

Applying the finite difference method to solve the parabolic equation (4) and (5) is the integration of the differential equation over a space S, for details see Smith [8]. One finite-difference approximation to equation (4) is

$$\frac{U_i^{j+1} - U_i^j}{k} = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} \qquad \text{--------} \quad (6)$$

Where

$x_i = i*h$, for $i = 0, 1, \ldots, n$;

$t_j = j*k$, for $j = 0, 1, \ldots$.

equation (6) can be written as

$$U_i^{j+1} = rU_{i-1}^j + (1-2r)U_i^j + rU_{i+1}^j \qquad \text{--------} \quad (7)$$

Where $r = \frac{k}{h^2}$,gives a formula for the unknown temperature, Ui at the (i, j+1) mesh point in terms of known " temperature " along the jth time row, as in Figure 4.

The analytical solution of the partial equation (4) satisfying the initial condition (1) and the boundary condition (5) is given as:

$$U(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sin \frac{1}{2} n \pi \right) (\sin n \pi x) (\exp(-n^2 \pi^2 t)) \quad \text{----(8)}$$

For more details see Smith [8].

II- Solar heated house

Consider the region illustrated in Fig.5. This figure represents a house heated by the sun from the top and sides and the temperature at the bottom remains fixed and equal to zero. The problem is how to compute the temperature distribution throughout the house.

The boundary condition is imposed according to the following **assumptions:**

1) The top heated to a constant temperature of $20^{\circ}C$ by the sun.
2) The bottom maintains a constant temperature of $0^{\circ}C$ from the earth.
3) The temperature of the vertical sides is proportional to their height:

U (x,y) = 20y / H. ------- (8)

Where

U (x, y) is the temperature at width x and height y, and H is height of the house.

So, the boundaries conditions are:

$$\left.\begin{aligned} &U(x_i, 0) = 0, \quad \text{for all } x_i \in (0, L); \\ &U(x_i, H) = 20, \text{ for all } x_i \in (0, L); \\ &U(0, y_j) = U(L, y_j) = 20 * y_j / H \end{aligned}\right\} \quad \text{--------} \ (9)$$

1- Simulation approach

We divide the x-axis and the y-axis into m and n intervals, respectively. So, the region is divided into m × n grid points. This scheme of grids provides coordinate points to measure the temperature at each point (x_i, y_j). Initially, we imposed zero temperature for all grid points inside the room. Next, we started to compute the temperature, U(x_i, y_j) , according to the random numbers generated. The temperature at the point (x_i, y_j) has been computed by observing the random path of a thermal messenger as it goes from (x_i , y_j) to some boundary point. The messenger bounces randomly from grid point to another grid point until it touches a boundary point. We call this messenger's path a random walk. When a path from the point (x_i, y_j) terminates at a boundary point B, the temperature at B is added to the temperature at (x_i, y_j). For each interior point (x_i, y_j), we perform k of those random walks, accumulating the heat U(x_i, y_j) and then we calculate the average of the accumulated temperature; U(x_i, y_j) / k.

2- Numerical approach

The domain of integration of a two-dimensional parabolic equation is always an area S bounded by a closed curve C. The problem can be written in non-dimensional form as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{-------} \ (10)$$

The simplest finite difference approximation is the five-point equation, which is based on the central difference as:

$$\frac{1}{k}(U_{i,j}^{n+1}-U_{i,j}^{n})=\frac{1}{h_1^2}(U_{i+1,j}^{n}-2U_{i,j}^{n}+U_{i-1,j}^{n})+\frac{1}{h_2^2}(U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}) \text{ ---- (11)}$$

Where k is the iteration factor, h1 and h2 are the interval space in x-axis and y-axis, respectively.

Let $r_1 = \frac{k}{h_1^2}$, and $r_2=\frac{k}{h_2^2}$, so the equation (7) become

$$U_{i,j}^{n+1} = r_1(U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}) + r_2(U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}) \text{ -----(12)}$$

For simplesty sake we suppose that $h_1 = h_2$, so $r_1= r_2 = r$. The equation (12) becomes:

$$U_{i,j}^{n+1} = (1-4r)U_{i,j}^{n} + r(U_{i+1,j}^{n} + U_{i-1,j}^{n} + U_{i,j+1}^{n} + U_{i,j-1}^{n}) \text{ ------ (13)}$$

The Result and Discussion

The results achieved through the present work discusses the two proposed problems. The results of the Monte Carlo simulation and the numerical finite difference are presented for both problems as the following:

I) Heated bar problem

The geometry of the problem studied and the boundary condition are shown in figure 1. The domain was chosen with bar length of unit (L = 1), and the number of grid points is **n** , the interval length is **h = 1/n** .

Table 1, shows the temperature distribution for simulation, numerical, and exact (equation 8) solutions beside the number of accumulated visits for each grid point and the computed probability to them. The computations are achieved for **n=10** and **nt=10000**. Table 2, shows the same information when **n=20**. The agreement is quite clear for both tables.

Figure 2, shows the exact solutions, simulation solutions , and numerical results . As expected the temperature distribution looks like as a normal distribution. The highest temperature occurred at mid point of the bar and the lowest temperature is at the end points. The simulation results achieved for **n = 10** and **nt = 10000**. While the numerical results was achieved when **r = 0.01**. This r value guaranteed the stability of the numerical results, for more details see Smith [8].

Figure 3, also shows the exact, simulation and numerical solutions when **n = 20** and **nt = 10000**. The numerical results achieved for **r = 0.005**.

The comparison of the results (exact, simulation and numerical) are shown good agreement, as it is clear in Figs. 2 and 3.

II) Solar House.

The geometry of the problem considered and the boundary condition is shown in figure 4. The computational domain was chosen with 5 units in the x-axis and y-axis, i.e. **width = hight = H = 5.** The number of grid points is **n = 5.** The interval length is

$$\frac{n}{H} = \frac{5}{5} = 1.$$

So, the house region is divided to 5×5 grid points.

Figure 5 shows the contour lines of the heat distribution of the solar house when n=5. The simulation results depicted in solid line while the numerical results are the doted line. The numerical results are performed for $r = 0.01$ of equation 13.

Figure 6, shows also the results of simulation and numerical approaches for n=10. The numerical results are achieved for $r = 0.005$. In both figures 5 and 6 show a good agreement for simulation and numerical results.

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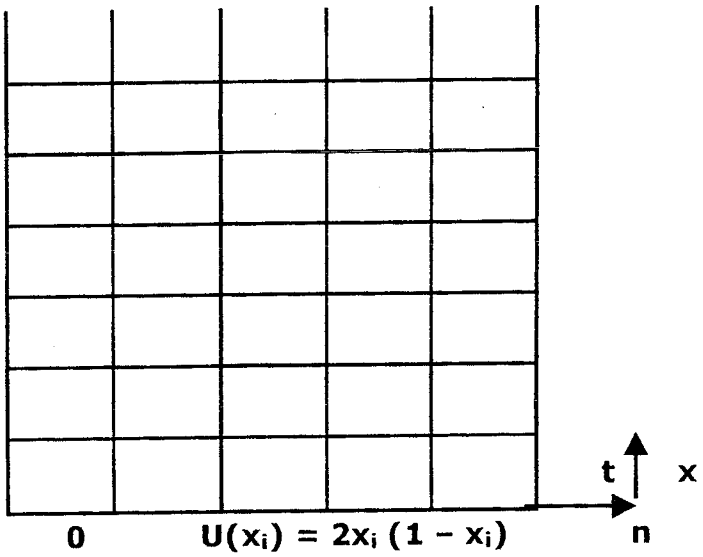

Fig. 1, The Boundary Condition of the Heated Bar.

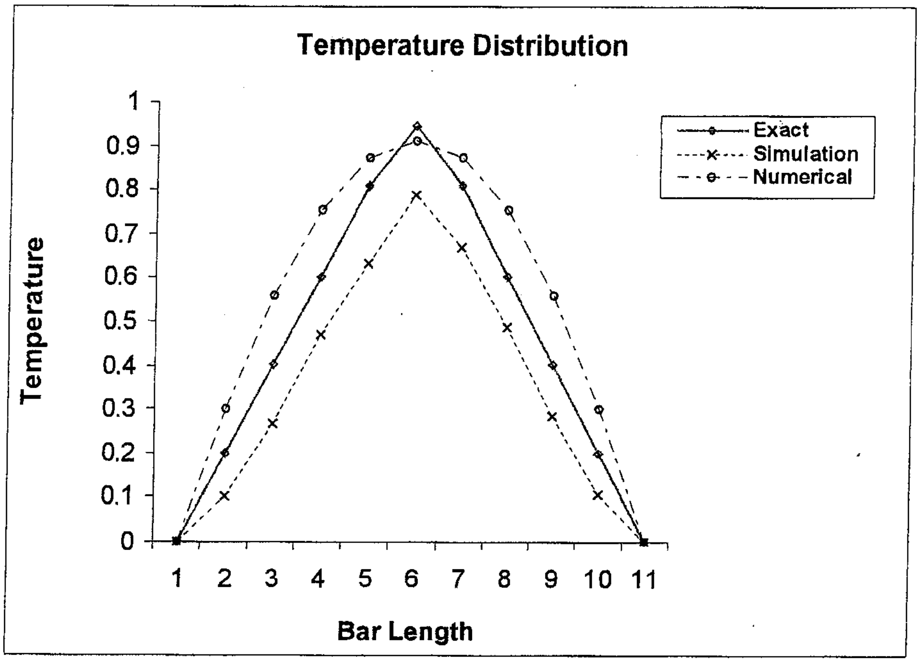

Fig. 2, Comparison Results of Heated Bar for n = 10.

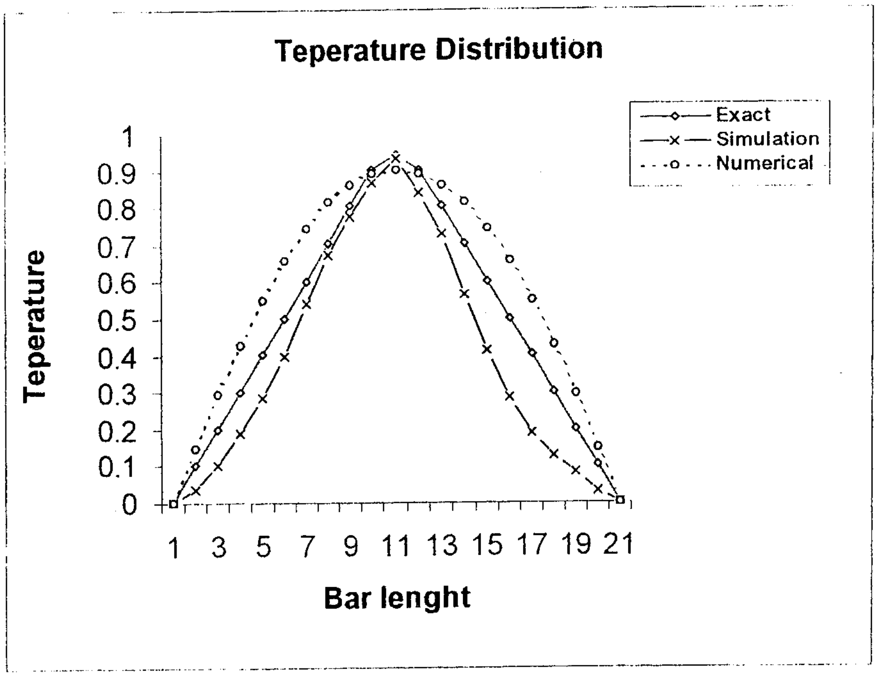

Fig. 3, Comparision Results of Heated Bar for n = 20.

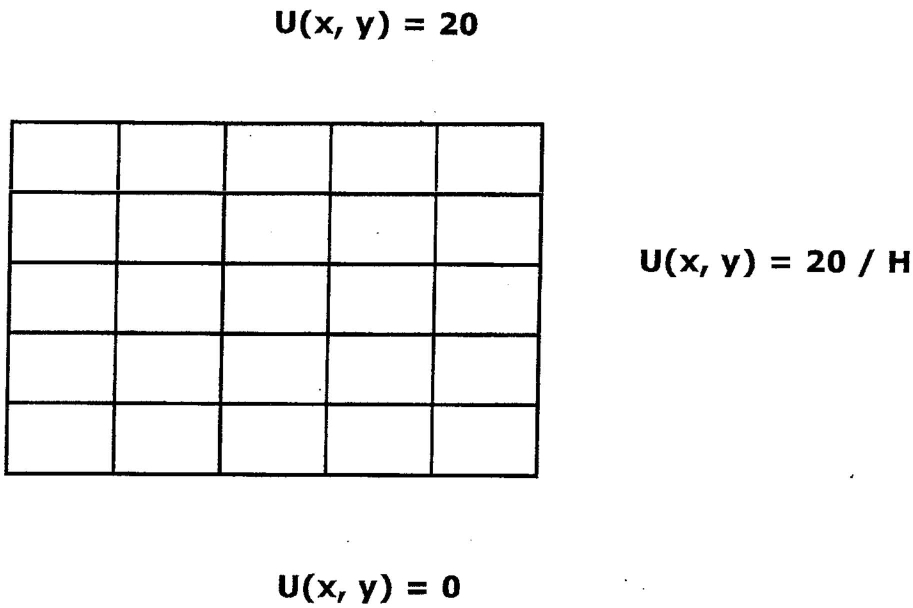

Fig. 4, The Boundary Condition of the Solar Heated House.

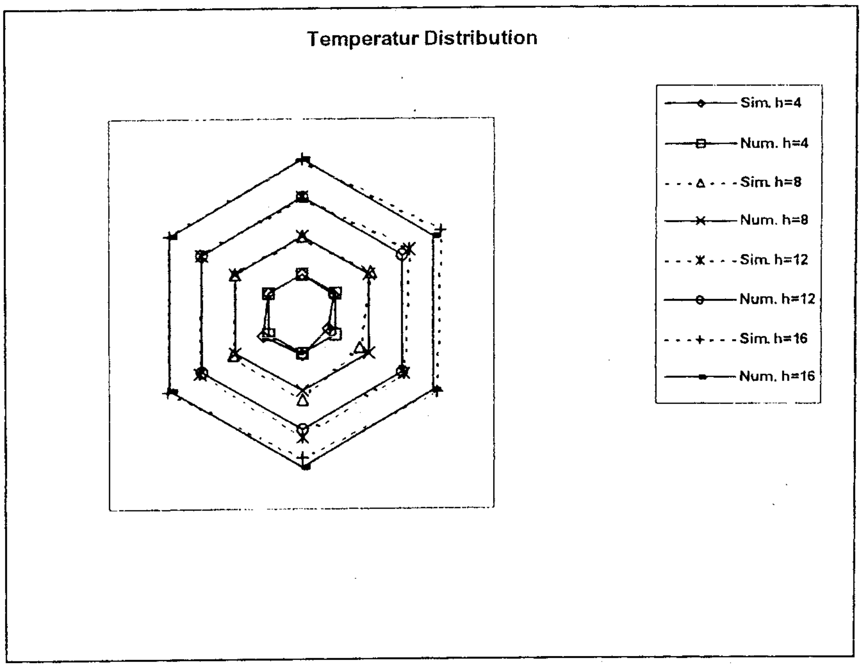

Fig.5, Solar Heated House, n=5

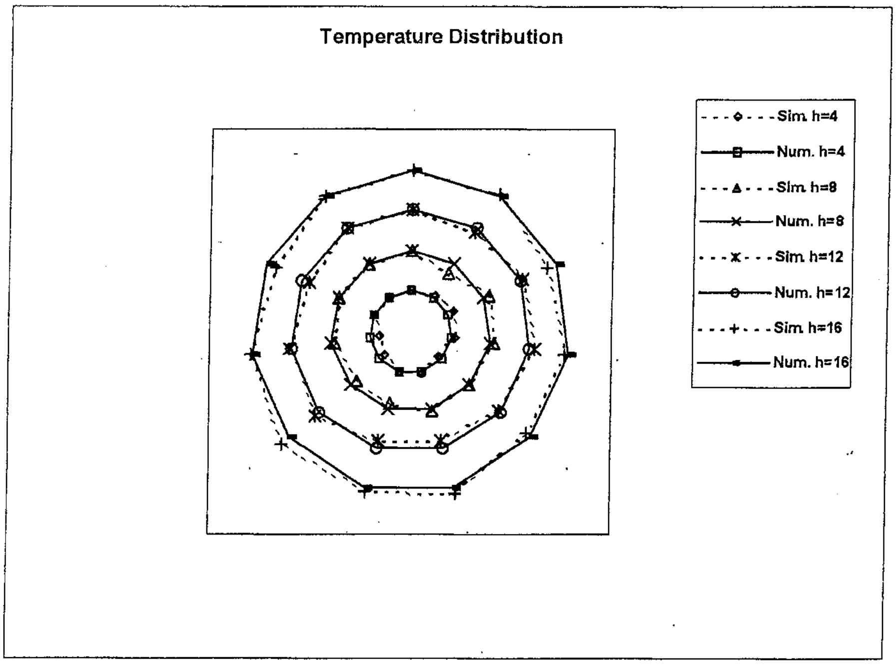

Fig. 6, Solar Heated House, n=10

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ملخص البحث :

تركز الاهتمام في هذا العمل على تطبيق أسلوب محاكاة الحاسب كوسيلة حل للمسائل. أُستخدم أسلوب مونت-كارلو لدراسة مسألة التوزيع الحراري لنوعين من المسائل. المسألة الأولى أحادية البعد "تسخين قضيب حديدي". المسألة الثانية ثنائية البعد "تسخين غرفة بواسطة الشمس". تم مقارنة نتائج محاكاة كلا المسألتين باستخدام أسلوب مونت- كارلو مع طرق الحلول العددية باستخدام أسلوب الفروقات المنتهية أحادية البعد وثنائية البعد ، على التوالي. لقد أظهرت نتائج المقارنة التوافق الكبير بين الأسلوبين (مونت- كارلو و الحل العددي) للمسألتين.